- These slides/notes represent only part of the course, and were accompanied by face-to-face explanations on white-board and additional topics / learning materials.
- In preparation of these slides I have also benefited from various books and online material.
- Some of the slides contain animations which may not be visible in pdf version.
- Corrections, comments, feedback may be sent to https://www.linkedin.com/in/naveedrazzaqbutt/

EE 322
Digital Communications

with

Dr. Naveed R. Butt

@

Jouf University

Introductions …

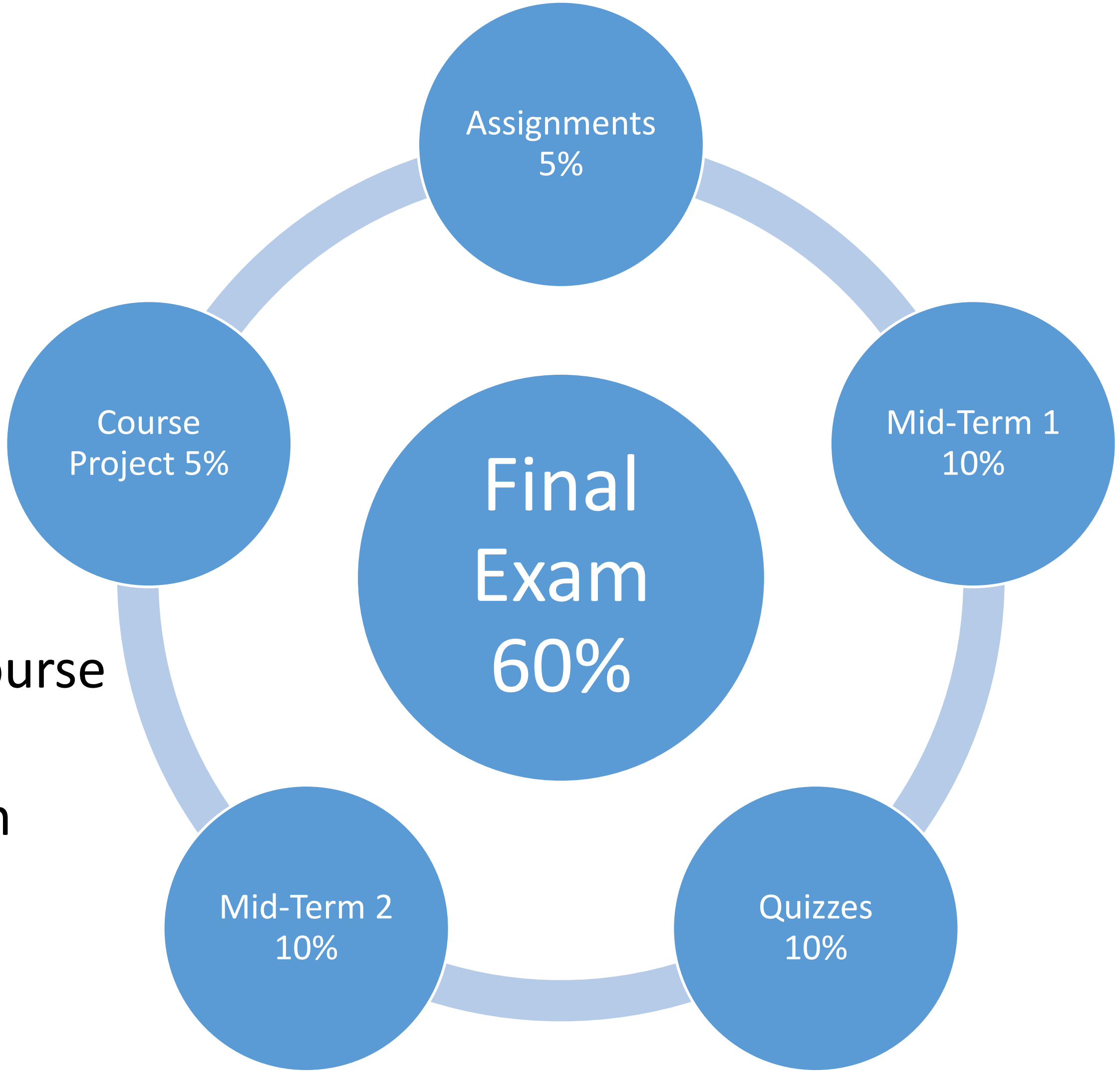
- Me
- You
- The Course

Important Business!!

- 75% attendance is mandatory!
- Textbooks
  - Simon Haykin, “Digital Communication Systems”, 1st Edition, 2013.
  - John Proakis and Masoud Salehi, “Digital Communications”, 5th Edition, 2007.
- Contact
  - nbutt@ju.edu.sa
  - office: 1140

Learning Plan

- **Lectures**
  - Help discover and grasp new concepts
- **Quizzes**
  - Help prepare/revise each week's concepts
  - Keep you from lagging behind in course
- **Assignments**
  - Learn to solve involved problems based on course
- **Course Project**
  - Helps learn independent work & presentation
  - Prepares for final year project
- **Exams**
  - Help prepare entire course material



What are some of our common daily activities?

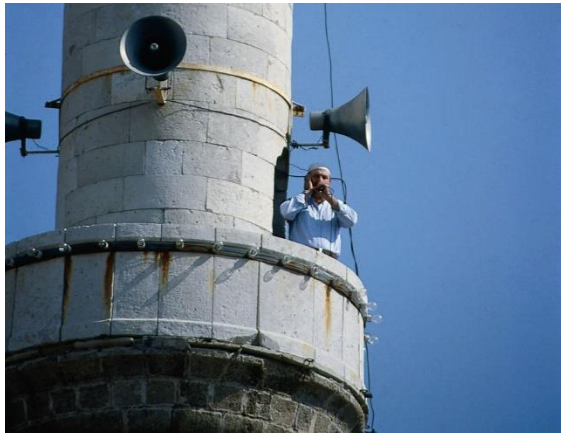
What are some of our common daily activities?

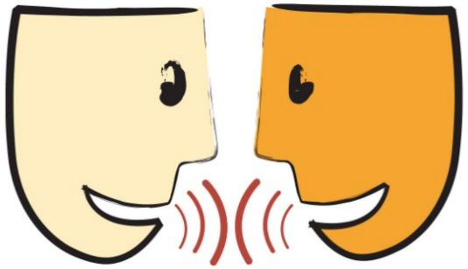
What are some of our common daily activities?

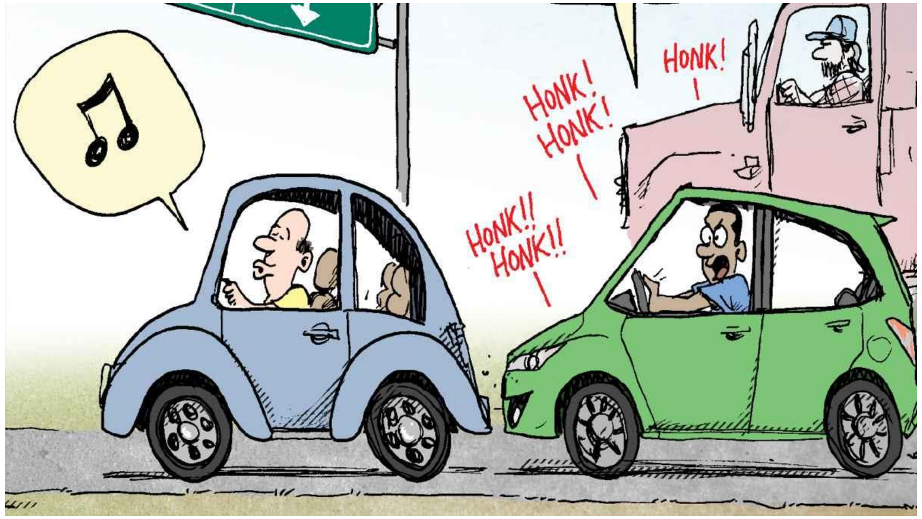
- Eating
- Drinking
- Sleeping
- ….
- …..
- Communicating!!

“Communicating” is an important human activity!!

“Communicating” is an important human activity!!





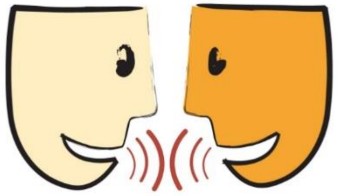


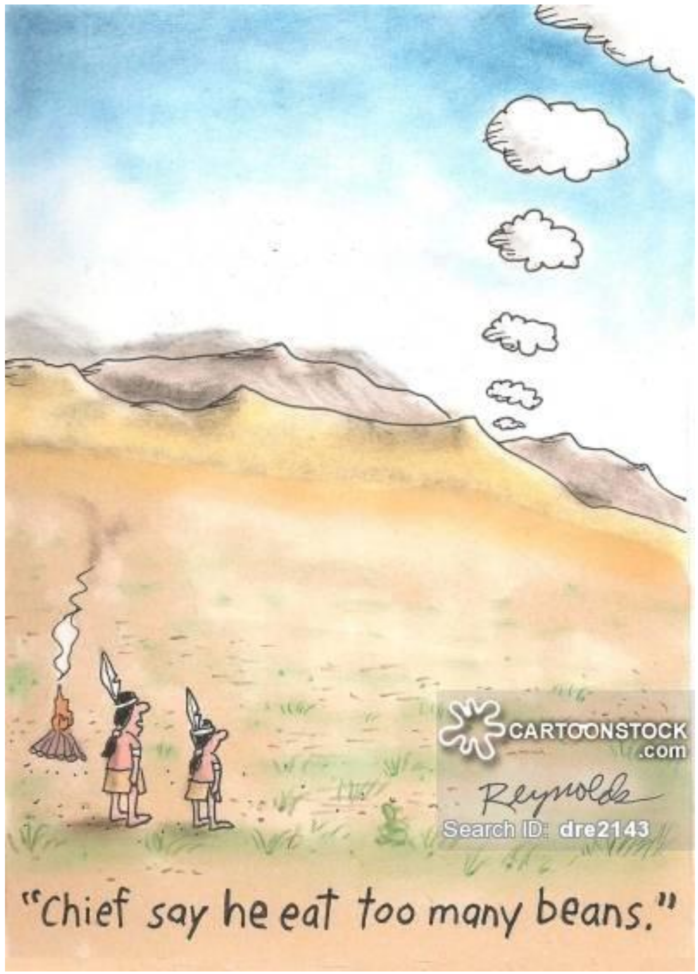
"Communicating" is an important human activity!!

- Instances of your communications today
  - Prayer call
  - Greeting family
  - Car honking
  - Car indicators
  - Waving to a friend
  - …
  - ….
  - Listening to this lecture…

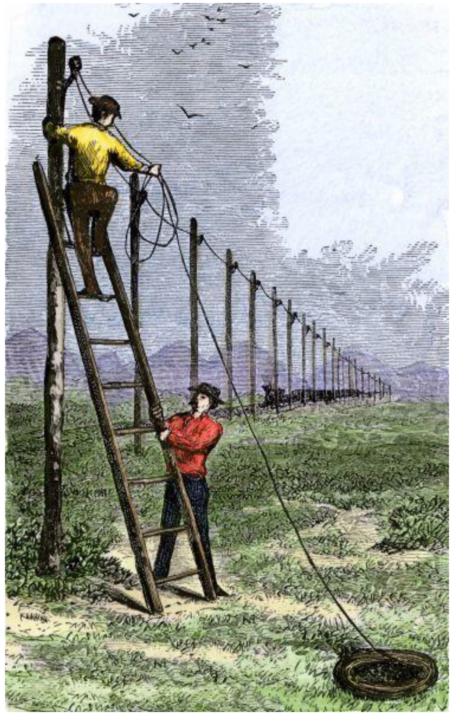
How have humans been communicating?

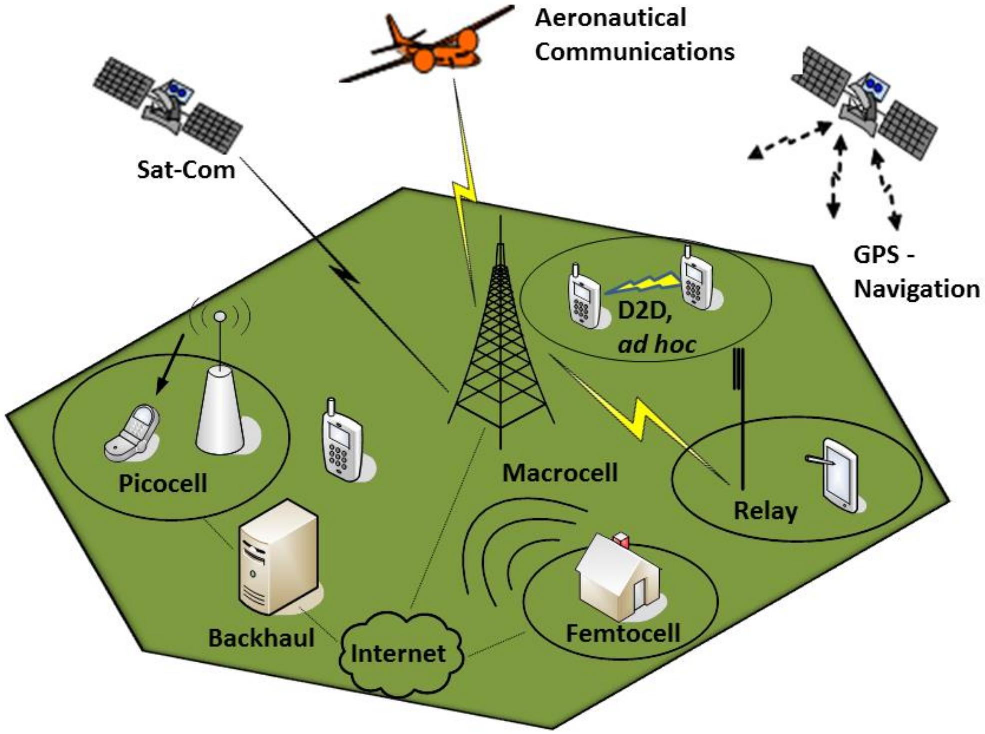
How have humans been communicating?





How have humans been communicating?





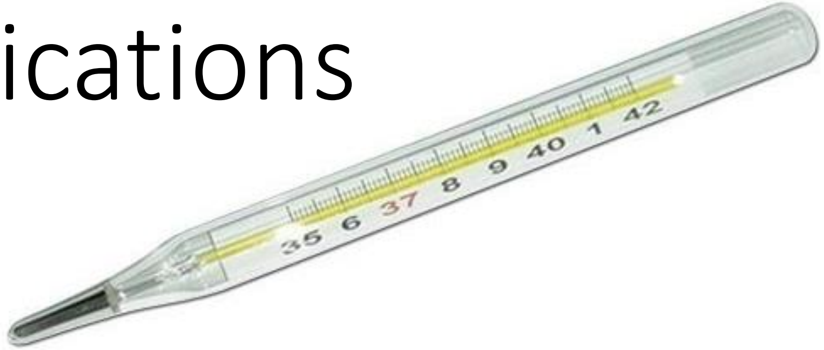
How have humans been communicating?

- Gestures (hand signs etc.)
- Verbal (talking)
- Drawing (cave walls, earth)
- Writing (clay, leaves, paper)
- Smoke/Gong Signals
- Mail (on foot, horses, ships, trains, planes, smaller vehicles)
- Electric Current (telegraph, landline telephone)
- Electromagnetic Waves (wireless communications)

What are the most distinct features of modern communications?

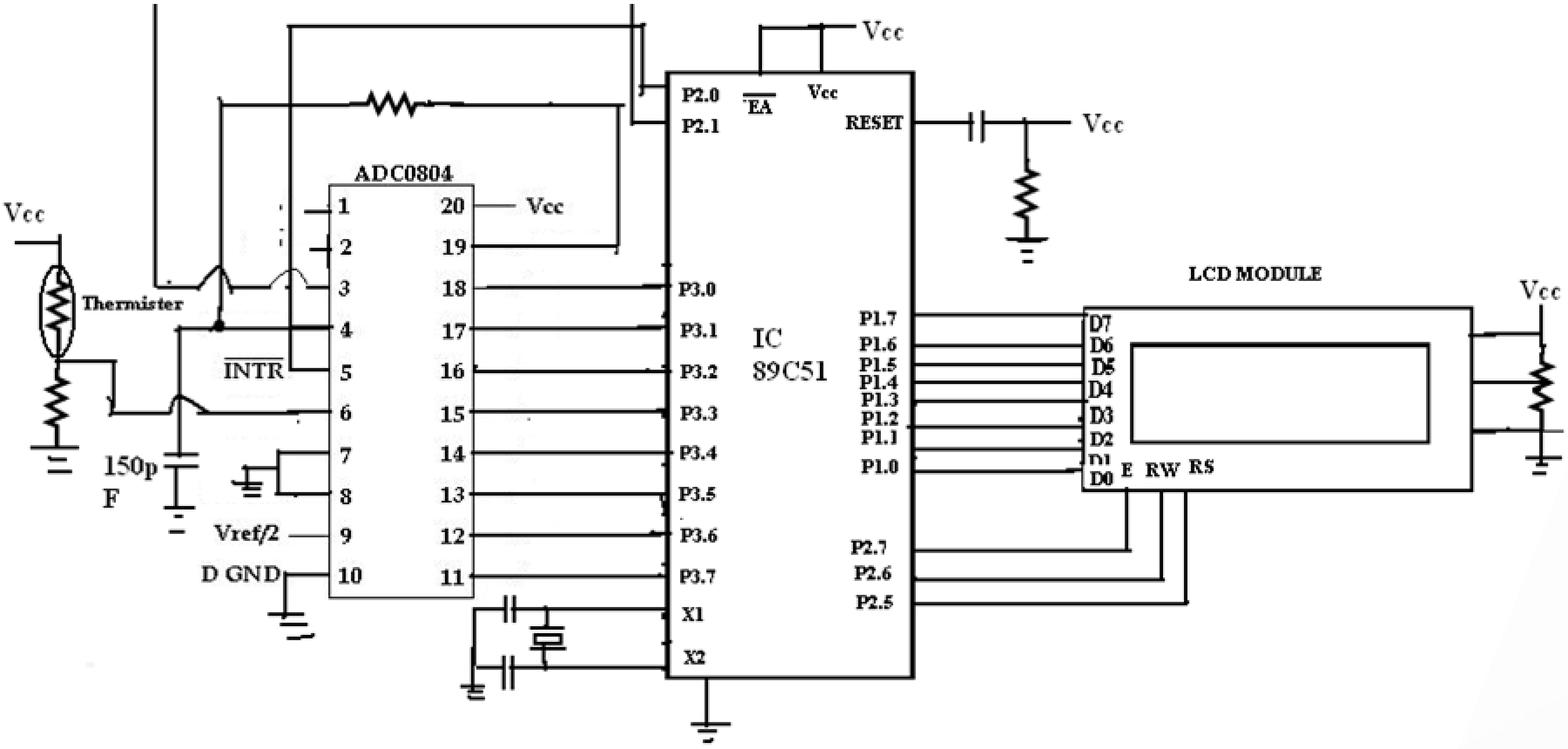
- Uses electric current or electromagnetic waves!
- Long distance
- Fast
- “Digital”

The "digital" in Digital Communications



- What is a "digit"?
  - Latin: Finger (used for counting)
  - Binary: 0 or 1 (ON or OFF)
  - Decimal: 0, 1, 2, 3, 4, 5, 6, 7, 8, 9
- Analog vs Digital
  - Analog
    - Continuous
  - Digital
    - Represented by fixed limited values (usually binary)
    - Typically involves electronic sensors, digitizers, digital logic.

The “digital” in Digital Communications



The “digital” in Digital Communications

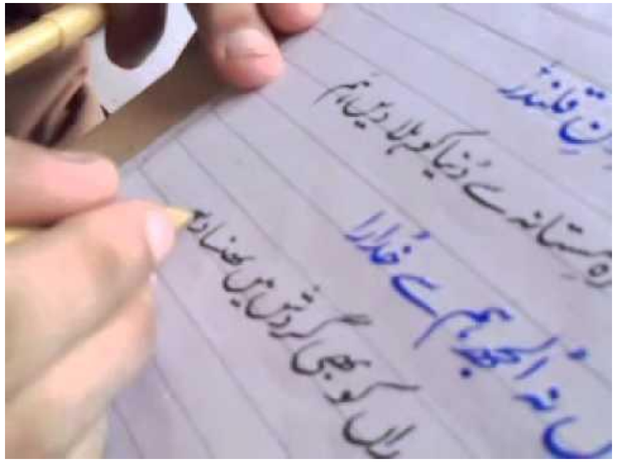
- Why go “digital”?
  - More robust to noise and interference
  - Easier and wider range of signal processing (using digital logic)
- Any downside?
  - Processing and digitization can take time
  - Typically requires more bandwidth compared to analog

Let us write a letter …

- The thoughts are coming from your mind, you are ***the source***
- The letter is meant for communicating your thoughts to someone - ***the target*** (desired user)

Let us write a letter …

- You use **symbols** (characters) to represent your thoughts
  - The desired user should be able to understand those symbols!!
- You use the postal service to arrange the **delivery** (sending/receiving) of your letter

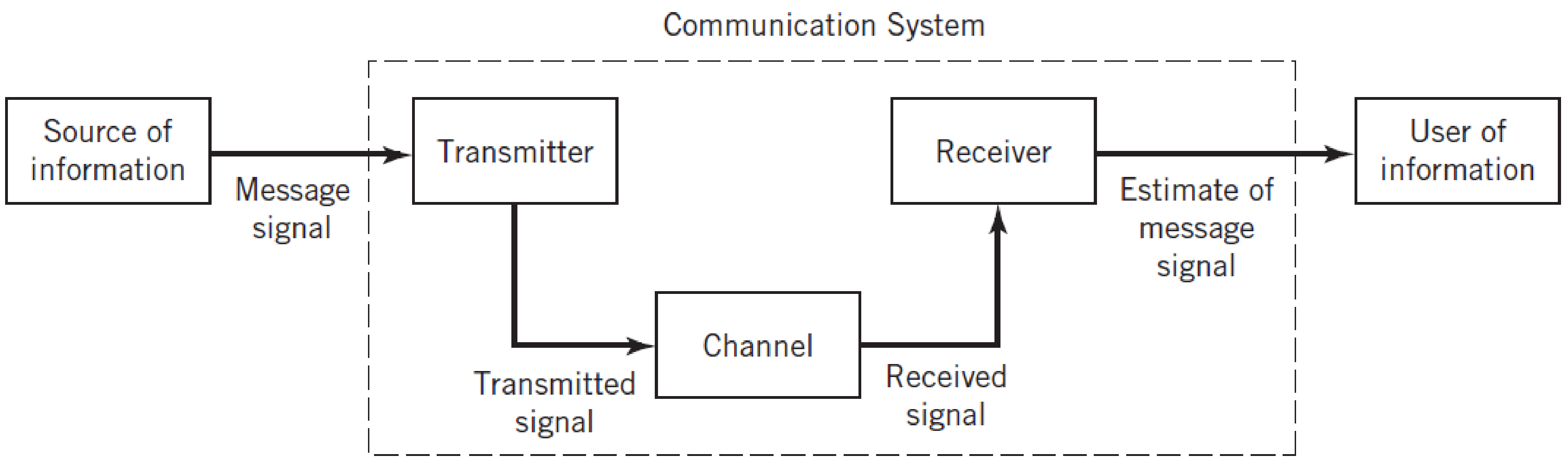


ا ب ت ث ج ح خ
د ذ ر ز س ش ص
ض ط ظ ع غ ف ق
ك ل م ن هـ و ي

Let us write a letter …

- The postal service uses some physical medium or ***"channels"*** (land/air) to transport your letter
- The postal service user various **containers that protect** the letters from dangers on the way (e.g. rain, getting torn or mixed up)

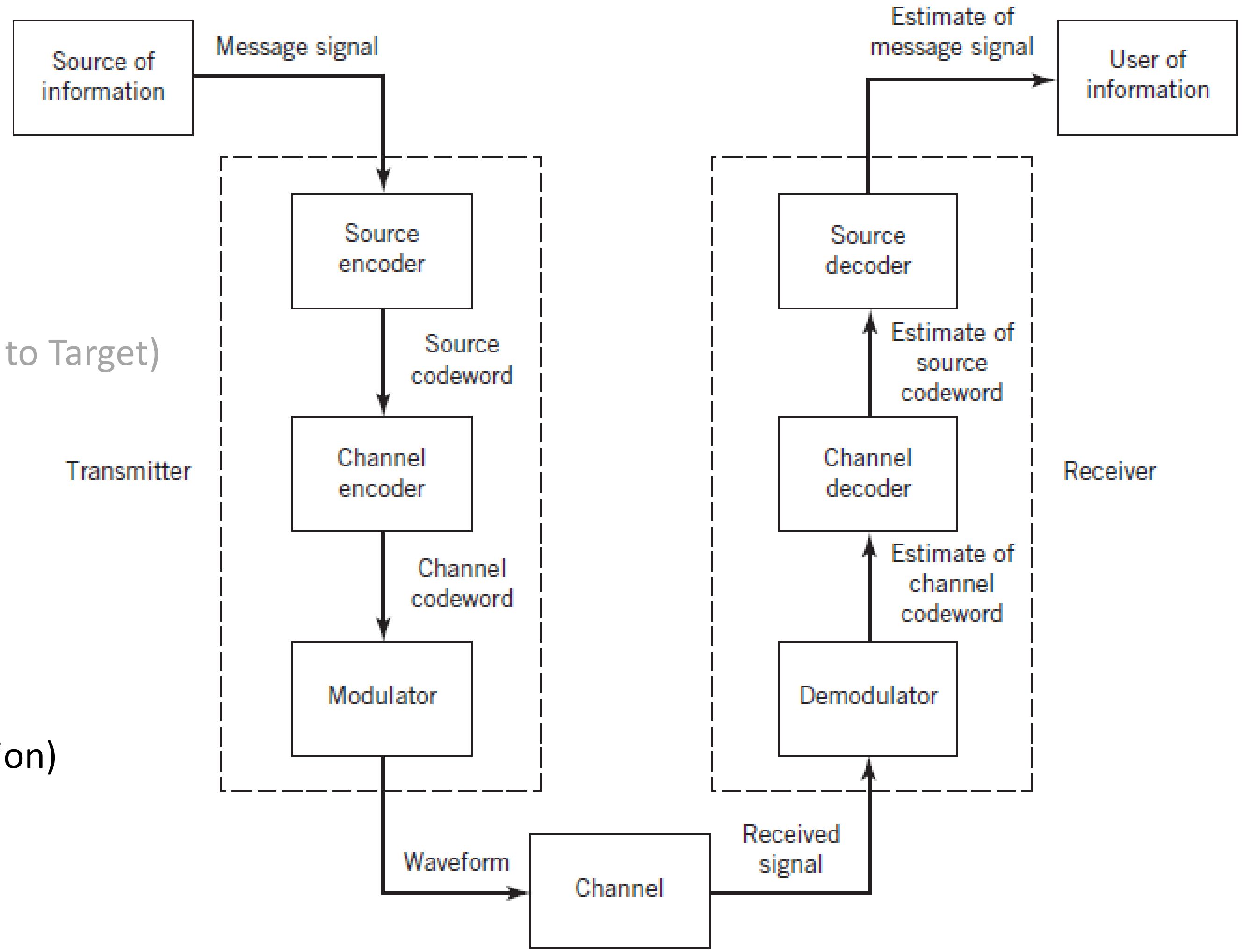
Elements of communication …



- **Source/Target**
  - (generator and desired user of information)
- **Transmitter/Receiver**
  - (mechanism to help deliver information from Source to Target)
- **Channel**
  - (physical medium used)

Elements of digital communication …

- **Source/Target**
  - (generator and desired user of information)
- **Transmitter/Receiver**
  - (mechanism to help deliver information from Source to Target)
- **Channel**
  - (physical medium used)
- **Modulation**
  - (analog symbols used for transmission)
- **Encoding**
  - (steps taken to protect the symbols during transmission)



Back to the letter …

- With your letter, what are some of the most desirable aspects?
  - **High Quality** - The message you wanted to convey should reach the target without changes
  - **High Speed** - Delivery should be fast
  - **Low Cost** - Delivery should cost as little as possible
  - **Security** – Your letter should not be given to wrong person

- *This is more or less what we want in digital communications as well!!*

Some common challenges in digital communications

- Channel Effects (noise, fading, scattering etc.)
- Processing Times
- Power Consumption
- Security and Privacy

In this course we will see …

- How to write **signals in mathematical form** (for analysis and design)
- What are the different **symbols we can use** for digital communications
- How can we **protect the signals** from channel effects
- What are some of the **ways of receiving** the transmitted signal
- What have scientists learned about the **transmission of information**

Questions?? Thoughts??

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Some familiar terms …

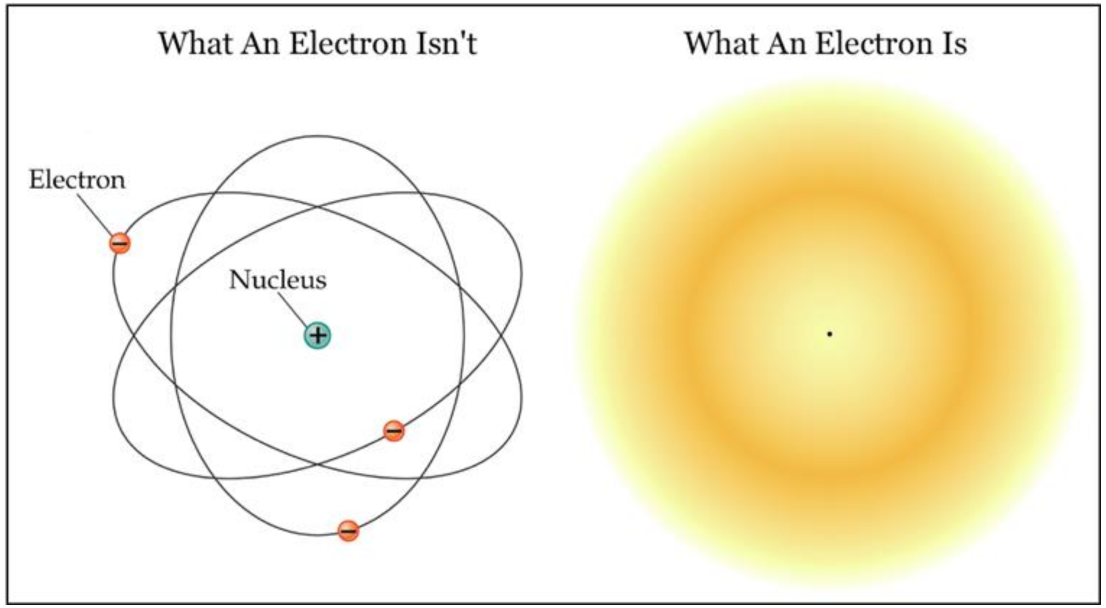
- Probability
- Gaussian Noise
- Fourier Transform
- Power Spectrum
- White Noise
- Linear Filters and Convolution

What is “probability”?

What is “probability”?

- Probability is a “lack of knowledge”!
  - We know you are here today. We are sure.
  - But will you be here in the next lecture? We are not sure anymore! There is now a “lack of knowledge”
    - “perhaps”, “maybe”, “probably”

Why do we sometimes lack knowledge?



Why do we sometimes lack knowledge?

- Future
  - A dice you haven’t rolled yet
    - How can we know which number it will show!
- Too hard to collect all the information
  - Which places did you visit today?
    - It may be possible to have a drone camera follow you all the time. Then we will not have “lack of knowledge” about places you go to. But this is too hard a thing to do.
- Quantum randomness
  - Where’s the electron?
    - According to current consensus, processes and properties at quantum level are probabilistic by their very nature.

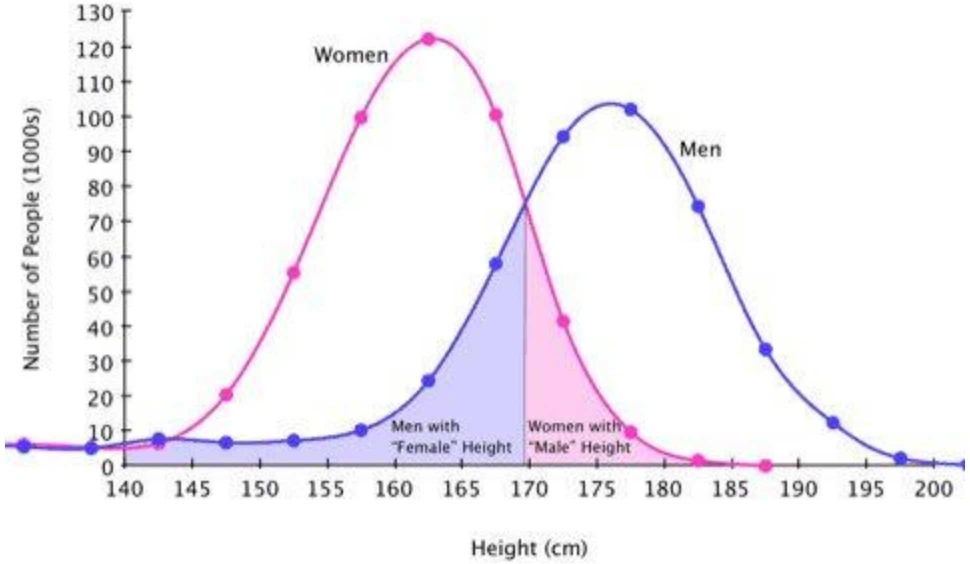
Why study probability?

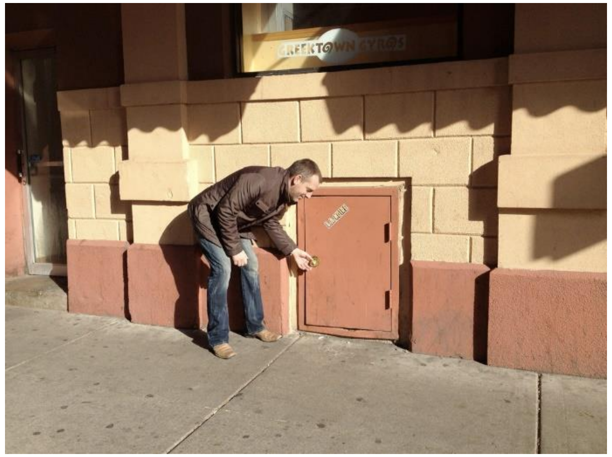
Why study probability?

- Height of the next student who enters the room.

Why study probability?

- Height of the next student who enters the room.





54 cm – 251 cm

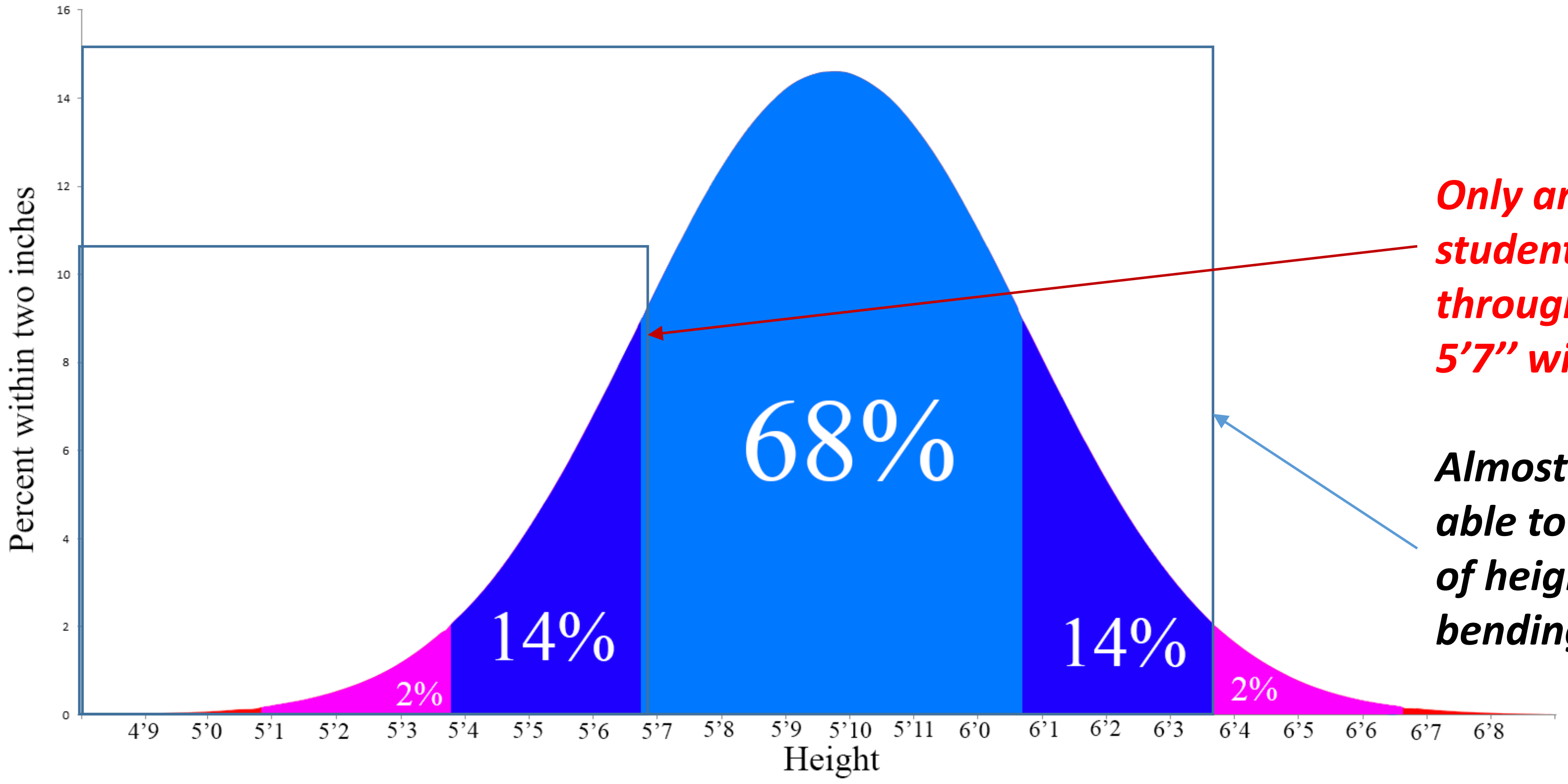
Why study probability?

- Height of the next student who enters the room.
  - There is lack of knowledge about it!
  - But that lack of knowledge is not "absolute"
    - We do *know something* about the heights of humans and can make some "guesses" based on whatever information we have (based on observation, experience, **statistics**)
  - Such guesses can help us design the height of the doorway (for instance).

Probability theory helps us make sense of an uncertain world!

- It helps us make ***smart guesses*** about uncertain events
- Based on the smart guesses we can ***plan, design, or take steps*** to better control the situation

Better design with probability!



Only around 16% of students will be able to pass through a door of height 5'7'' without bending.

Almost 98% students will be able to pass through a door of height 6'4'' without bending

“I hate this course”

“I hate this course”

- Suppose we record your voice saying this three times.
  - The three recordings will not be exactly the same
  - But we are still able to hear them and say that they belong to same person and what the person is saying
  - This means that there are some “common” or “fixed” underlying features in the recordings!
- Probability theory helps us understand the fixed aspects (mean, variance, covariance etc.) of uncertain processes.
- This “understanding” helps us design efficient systems
  - E.g. Speech and voice recognition for Siri, Google Assistant etc.

Using probability in communication

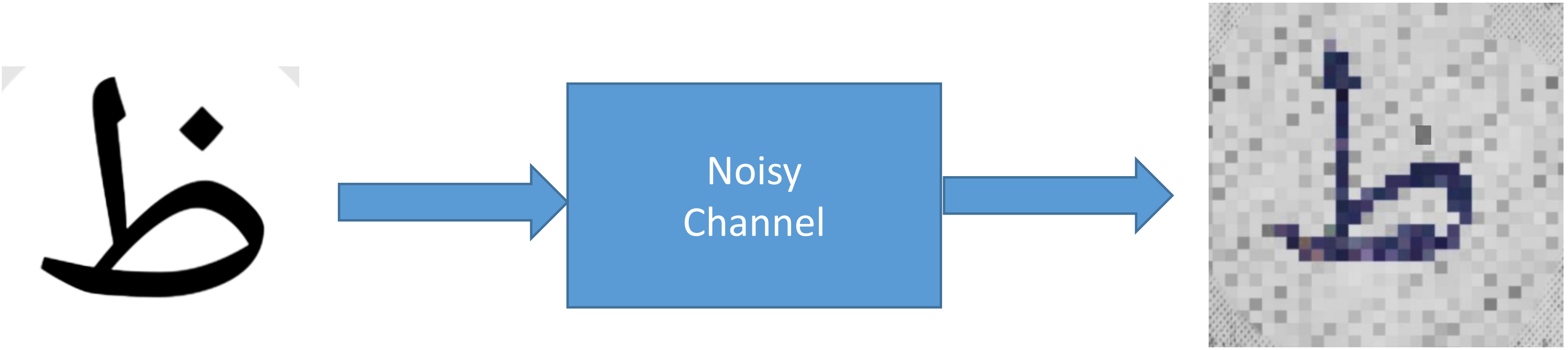
Probability of receiving “ta” when “za” is sent

P (ط | ظ) = High!!

P (ط | ع) = Low!!

Lesson: perhaps we should not use symbols that are too similar (close in some sense) to each other!

Using probability in communication



Some familiar terms …

- Probability
- Gaussian Noise
- Fourier Transform
- Power Spectrum
- White Noise
- Linear Filters and Convolution

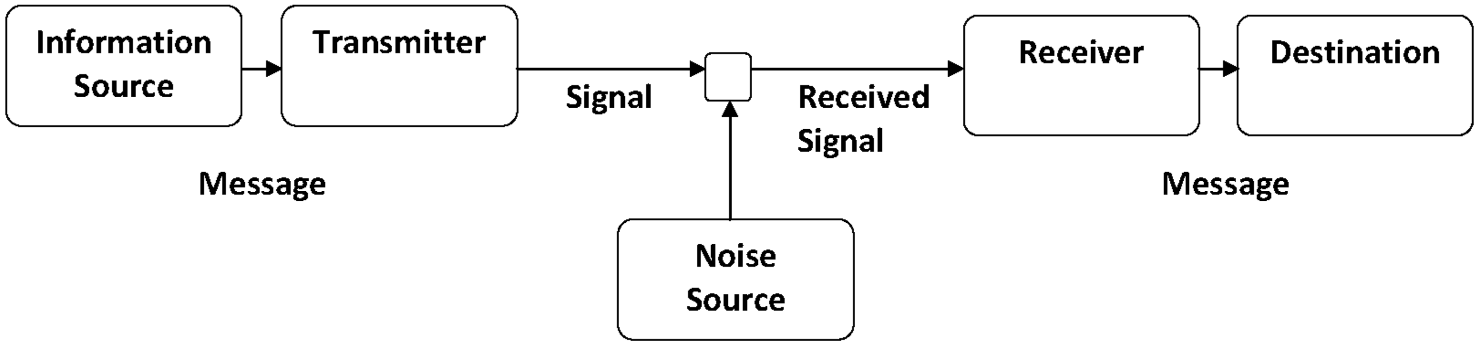
Gaussian Noise = a noise process that follows the Gaussian distribution

- Noise?
  - Something unwanted!
    - You are talking to your friend, cars are passing in the background, the traffic sounds are “noise”.

Gaussian Noise = a noise process that follows the Gaussian distribution

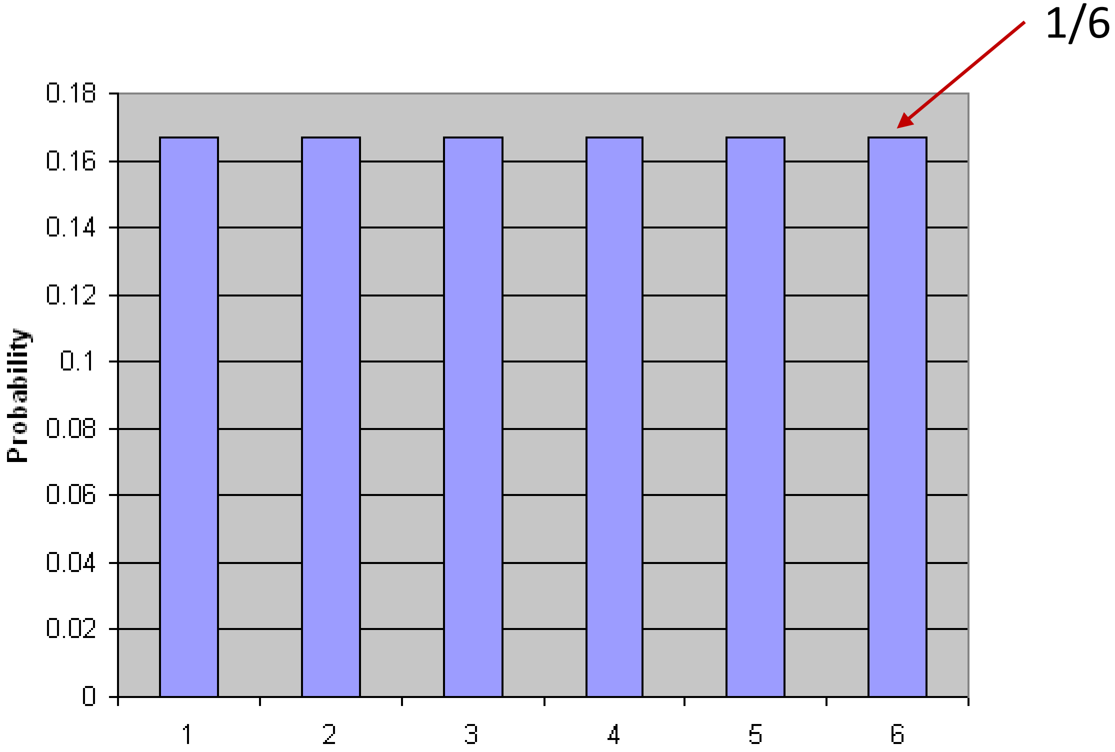
- Noise?
  - In digital communications, we usually represent ***unwanted effects or signals*** that show up during communication as "noise"
  - Our typical goal in digital communications is to ***reduce the effects of noise***

Gaussian Noise = a noise process that follows the Gaussian distribution



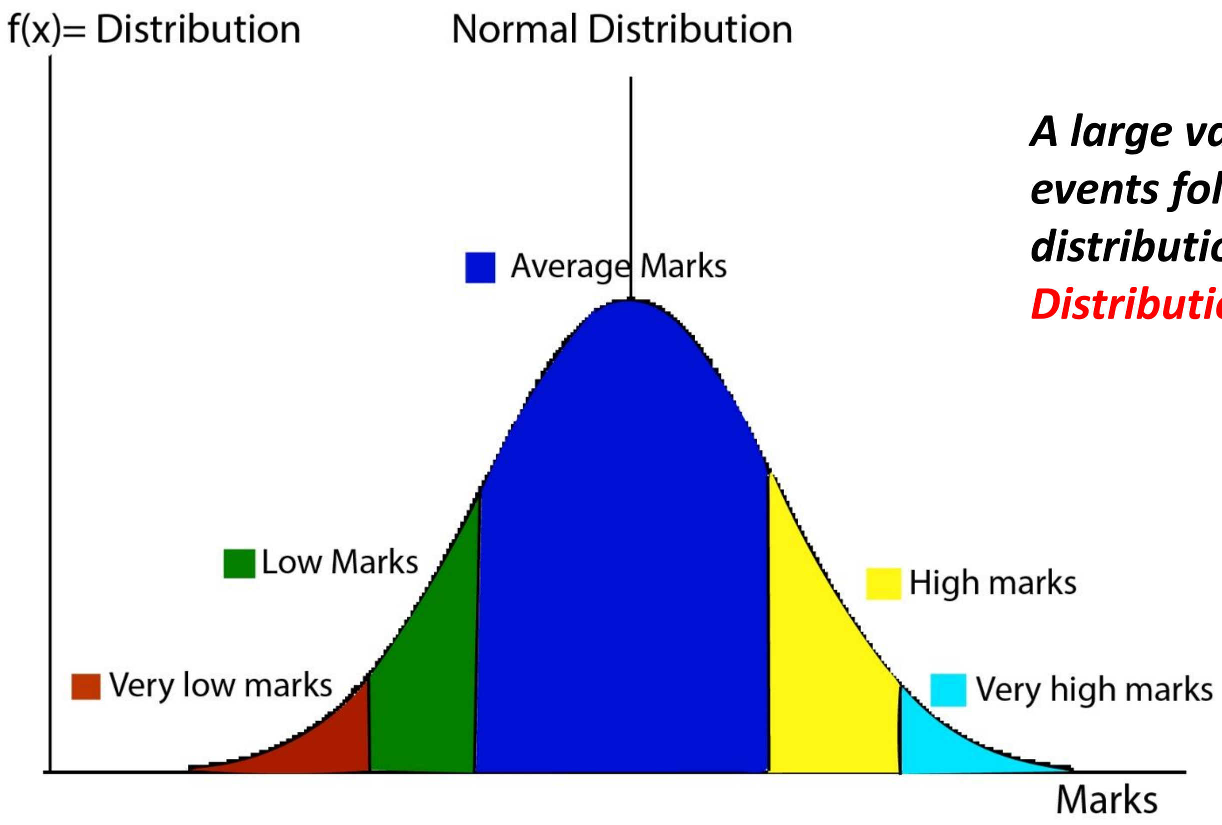
Gaussian Noise = a noise process that follows the Gaussian distribution

- Distribution?
  - A distribution is a collection of probabilities we assign to random events.
    - Collection? Graph, table, function



Gaussian Noise = a noise process that follows the Gaussian distribution

- Gaussian?



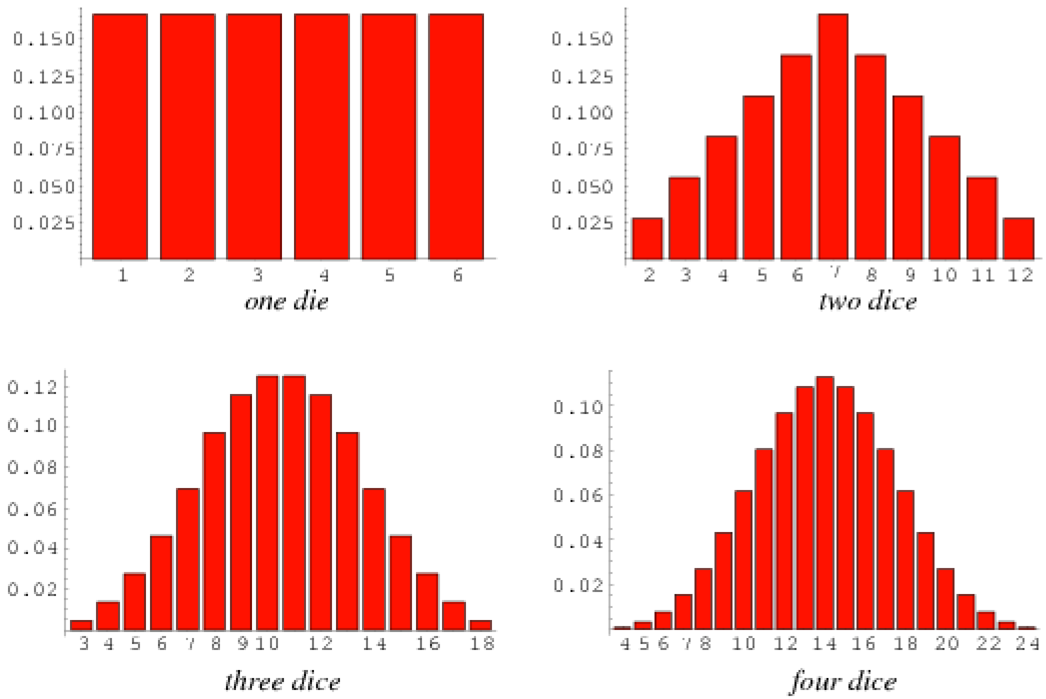
A large variety of random events follow a "bell" shaped distribution called the Gaussian Distribution

Why Gaussian?

- ***Common***
  - Random events often follow the Gaussian distribution
    - E.g., heights, ages, marks etc.

Why Gaussian?

- ***Central Limit Theorem***
  - The collective effect a large variety of random events is often Gaussian!



Uniform distribution slowly becoming Gaussian!

Why Gaussian?

- ***Convenience***
  - It is easy to model and work with (often a good "simplification")
    - Completely specified by mean and variance (or covariance)

What/Who is Gauss anyway?

Considered by many to be the greatest mathematician ever!!

(Germany 1777 - 1855)

Some familiar terms …

- Probability
- Gaussian Noise
- Fourier Transform
- Power Spectrum
- White Noise
- Linear Filters and Convolution

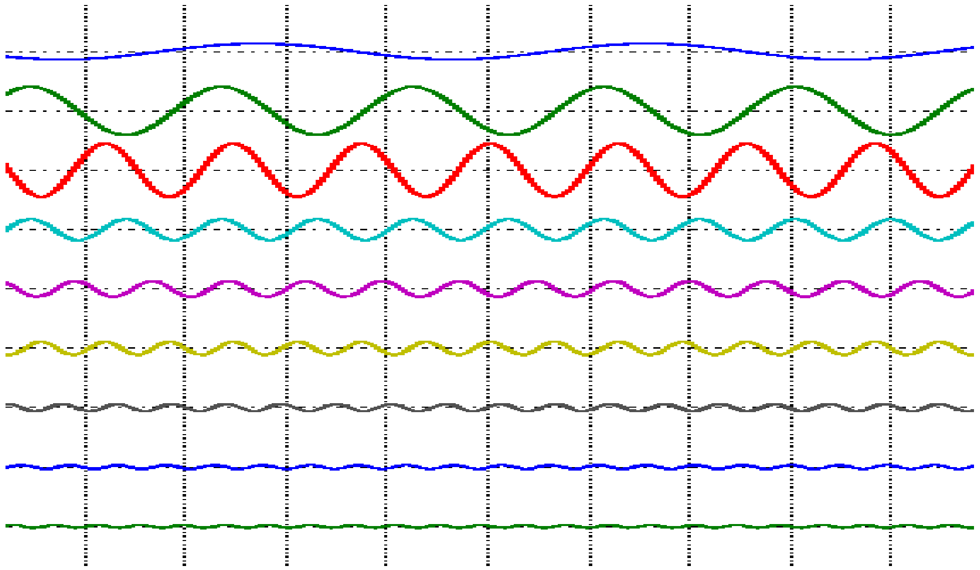
Let us bake a cake ...

- Suppose you are a great chef who always likes to see baked things in term of their ingredients

Ingredient	Amount	Process
Flour	1 cup	**Mix flour and milk ....**
Milk	1 cup	
Sugar	0.25 cup	
Egg	3	

Fourier Cake!

- In Fourier Transform, we want to look at signals in terms of a fixed set of ingredients
  - Ingredients : Sinusoids of different frequencies



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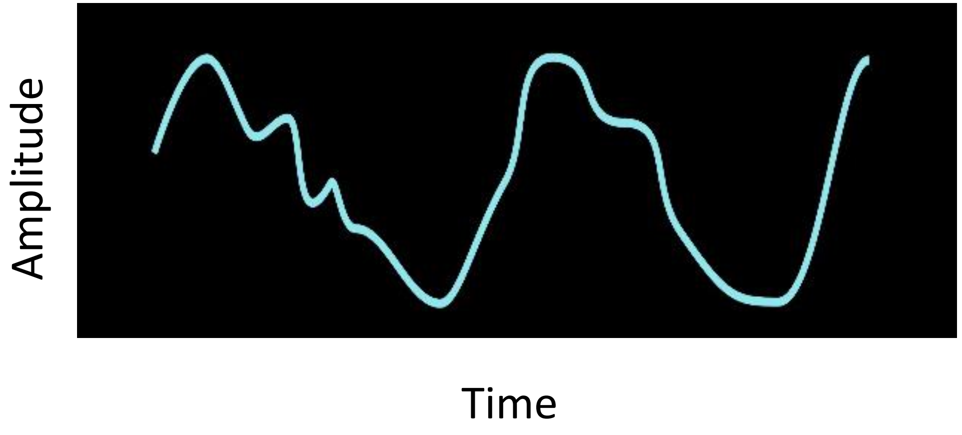

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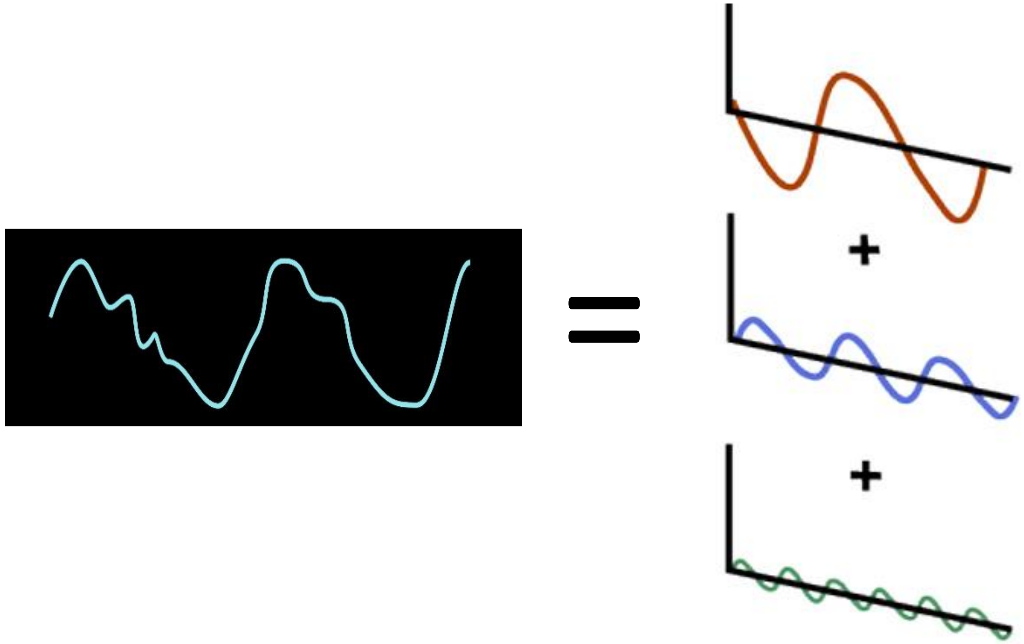
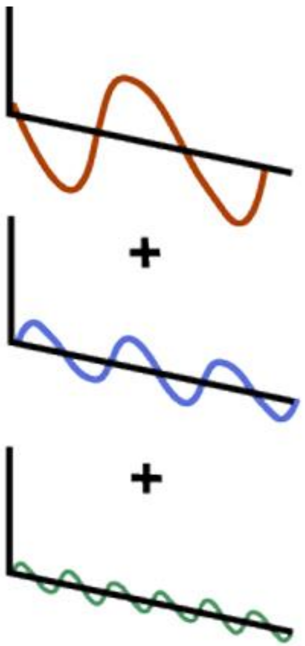


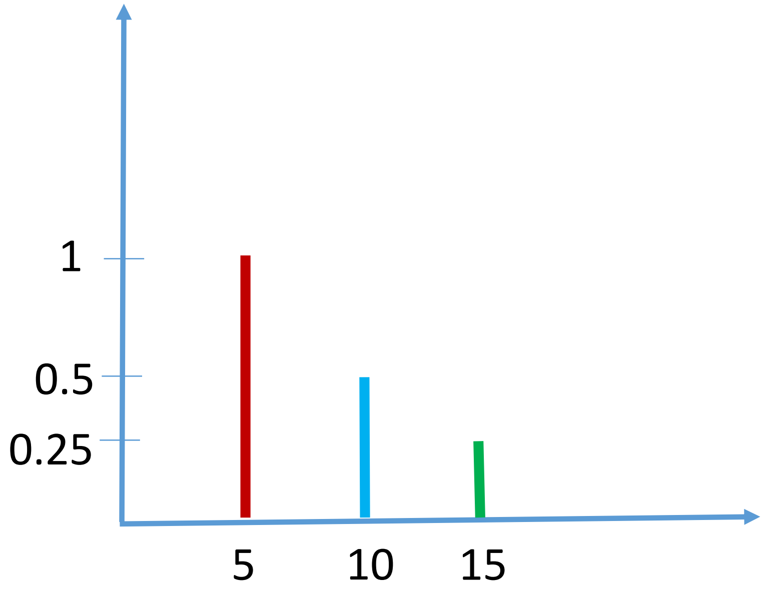
Ingredient (sinusoid frequency)	Amount (scaling)	Process
f_1	1	**Add all**
f_2	0.5	
f_3	0.25	

Fourier Transform

- How is this shown after Fourier transform?



Ingredient (sinusoid frequency)	Amount (scaling)	Process
5 Hz	1	**Add all**
10 Hz	0.5	
15 Hz	0.25	



Fourier Transform

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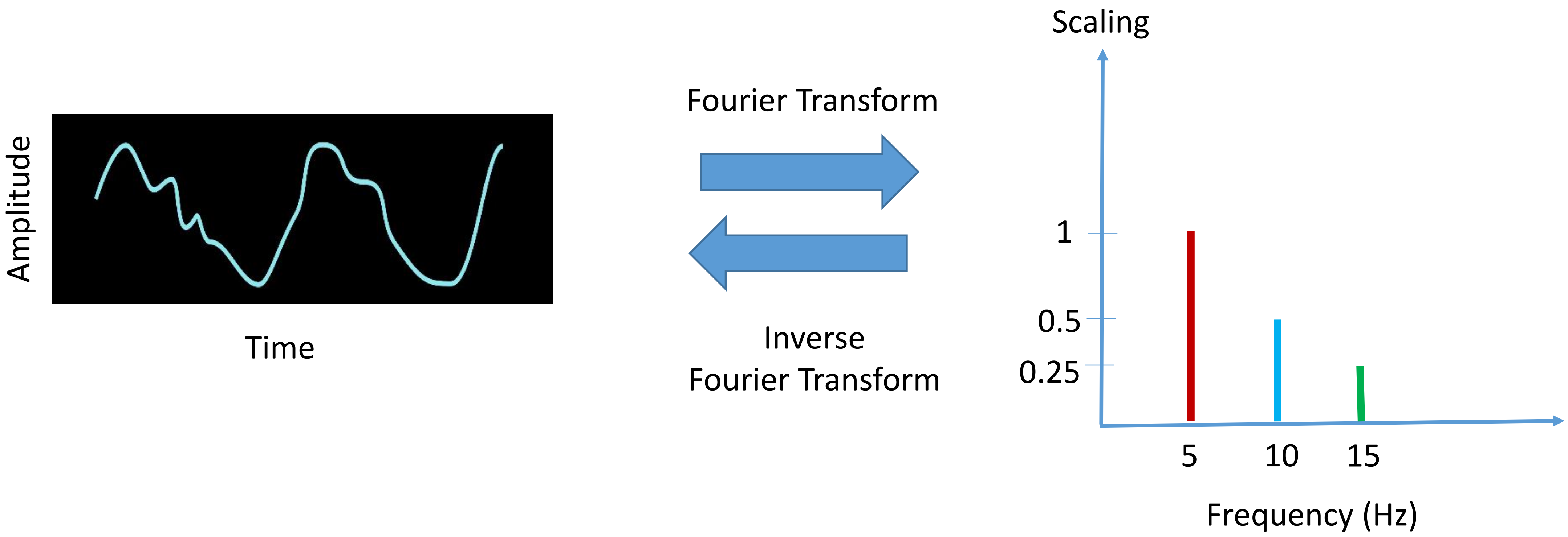
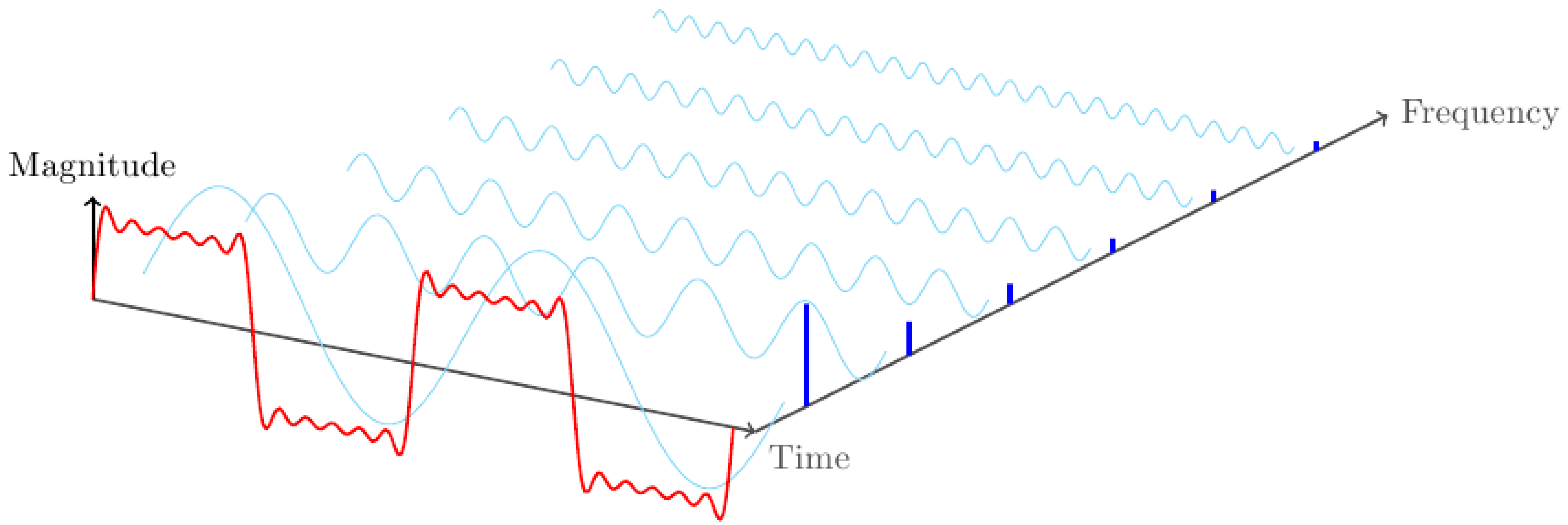
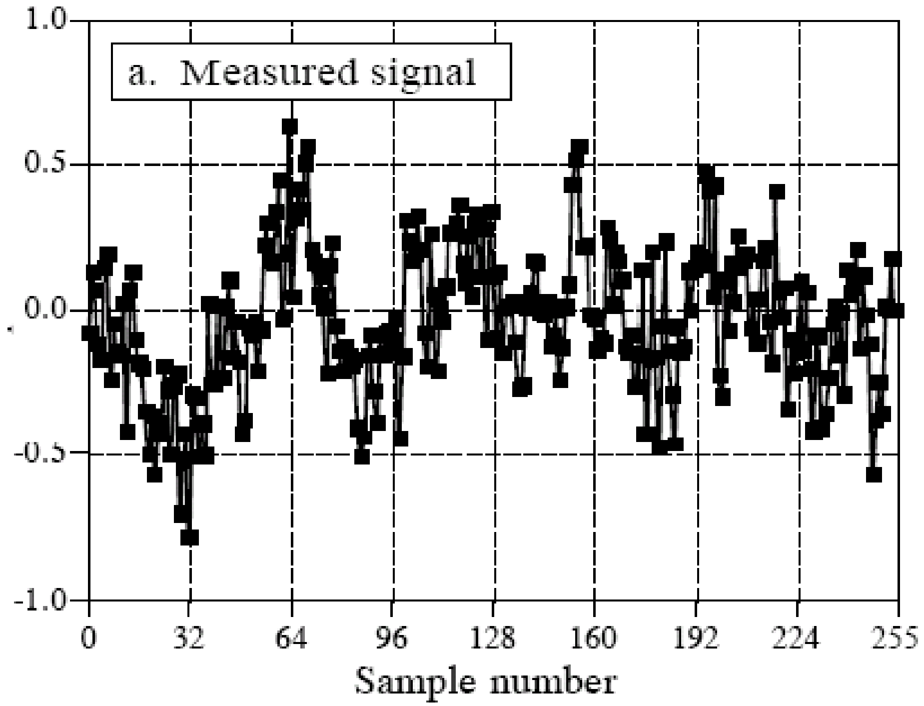


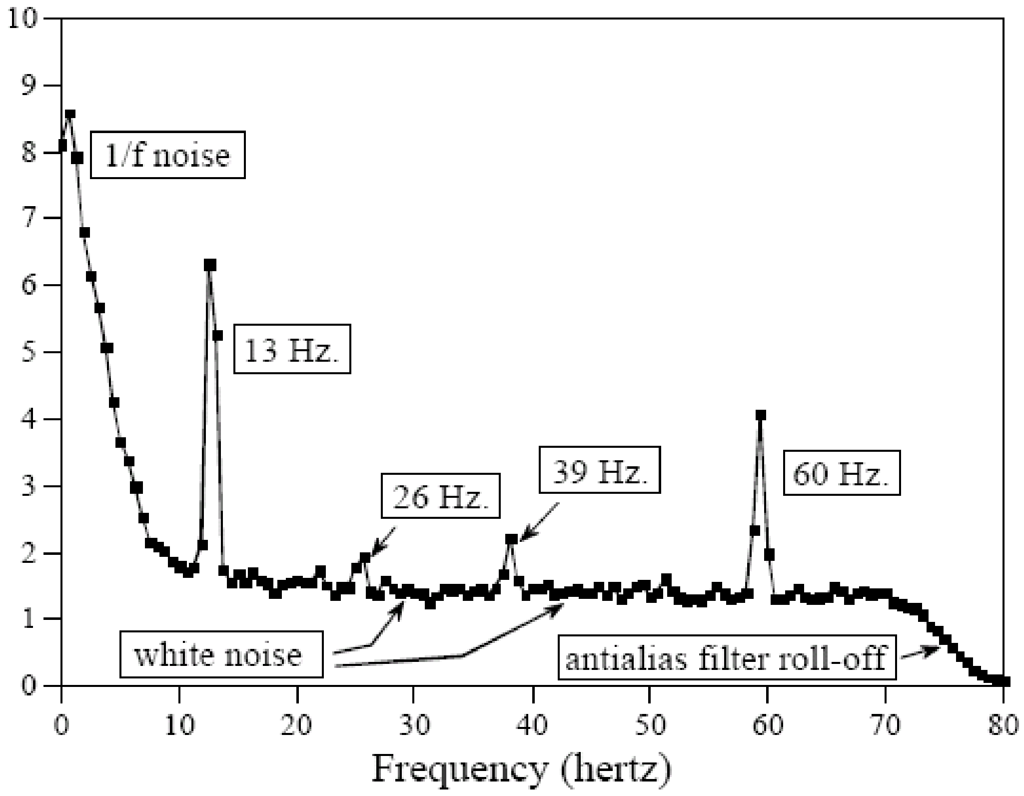
Fourier Transform

- Another example



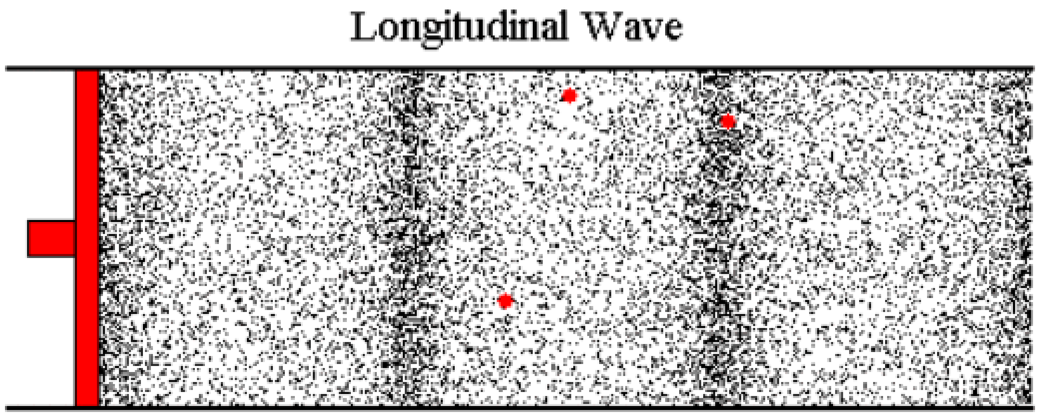
Why transform from time to frequency?

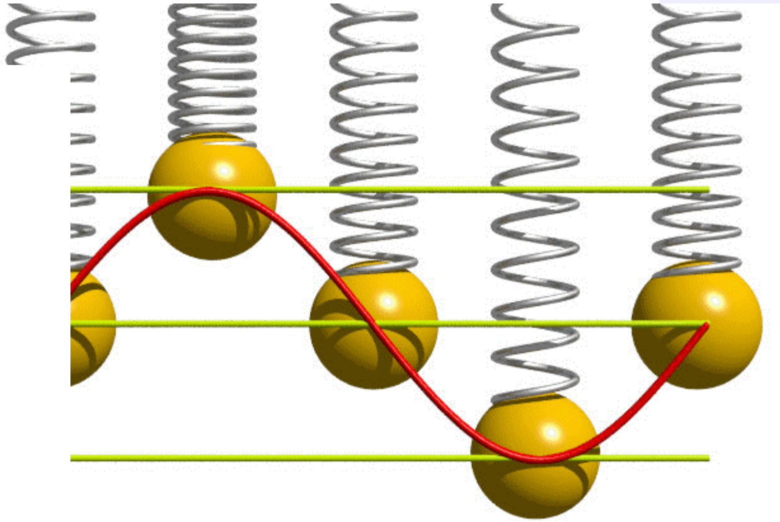




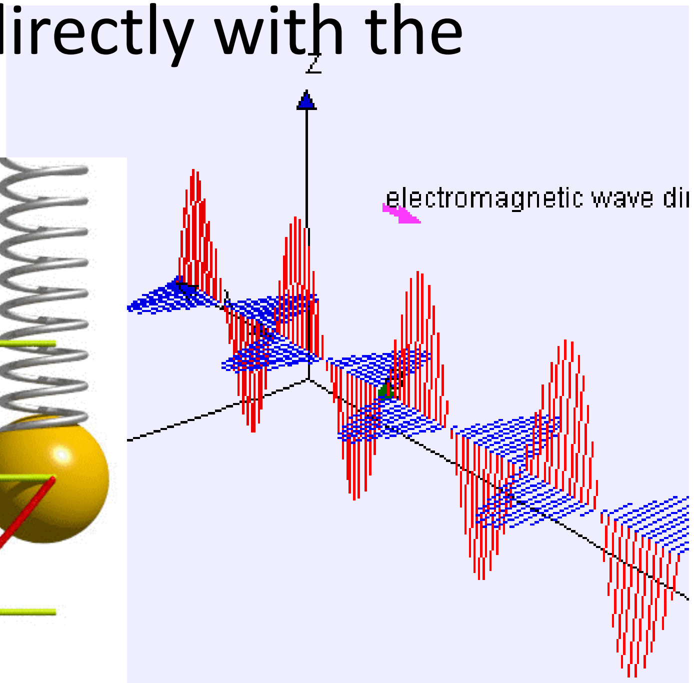
Why transform from time to frequency?

- Many natural processes (e.g. vibration, rotation) are periodic (cyclic) in nature. It is good to represent them by frequency.
    - Examples: sound waves, electromagnetic waves.
- By writing signals in frequency domain, we can play directly with the frequencies.





Friedrich A.Lohmüller, 2012



Why sinusoids?

- Sinusoids provide a good model for vibration and rotation
- Sinusoids are smooth functions and are mathematically easier to work with
  - E.g., derivative of a sinusoid is also a sinusoid, and integral of a sinusoid is also a sinusoid

Questions?? Thoughts??

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In the last lecture we saw that…

- Probability is a lack of knowledge.
- Randomly varying or uncertain events may still have some underlying characteristics that are “fixed”
- Some of these can be

Probability distribution ← If this does not change over time then the random process is **strictly stationary**

Mean, Variance, Covariance ← If these do not “vary” over time then the random process is **weakly stationary**

In the last lecture we saw that…

- Probability is a lack of knowledge.
- Random or uncertain events may still have some underlying characteristics that are "fixed"
- Some of these can be

If this does not change over time then the random process is **strictly stationary**

If these do not "vary" over time then the random process is **weakly stationary**

Today's topics

But first …

- Let us be clear about what we mean by a random "process"
- A random process is a **series of random events**
  - Can be discrete (student heights) or continuous (voice) or discretized (digitized voice)

Random Event/Variable → X

Height of next student who enters the room

Random Process (discrete) → $[X_1 \; X_2 \; X_3 \; X_4 \; X_5 \; X_6 \; X_7 \; X_8]$

Heights of next eight students who enter the room

But first …

- Let us be clear about what we mean by a random "process"
- A random process is a **series of random events**
  - Can be discrete (student heights) or continuous (voice) or discretized (discretized voice)

$X(t)$ *Human speech*

Random Process (continuous)

$[X(t_1)\ X(t_2)\ X(t_3)\ X(t_4)\ X(t_5)\ X(t_6)\ X(t_7)\ X(t_8)]$ *Samples taken at times t1, t2, …, t8.*

Random Process (discrete)

Our friends in an uncertain world!

- Random or uncertain events may still have some underlying characteristics that are "fixed"
- Some of these can be

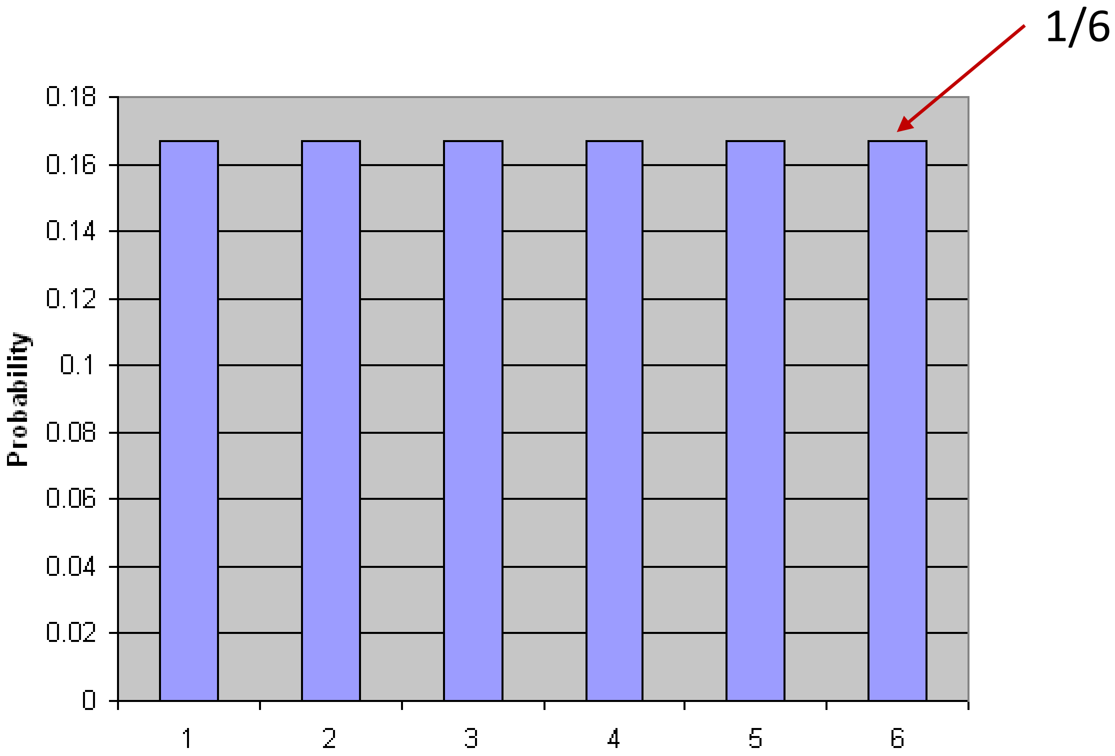
Probability distribution ← If this does not change over time then the random process is **strictly stationary**

↓

Mean, Variance, Covariance ← If these do not "vary" over time then the random process is **weakly stationary**

Distribution?

- A distribution is a collection of probabilities we assign to random events.
  - Collection? Graph, table, function



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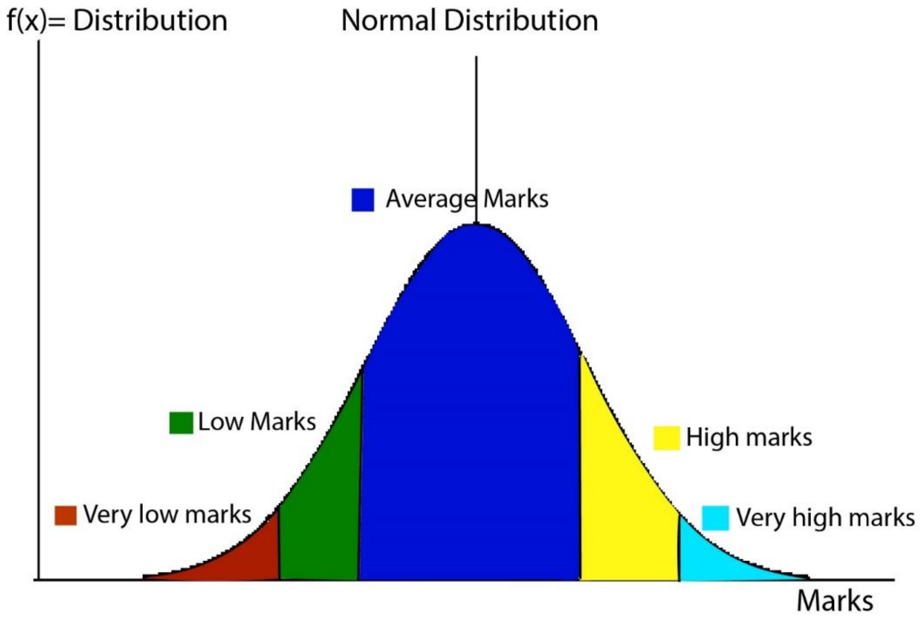
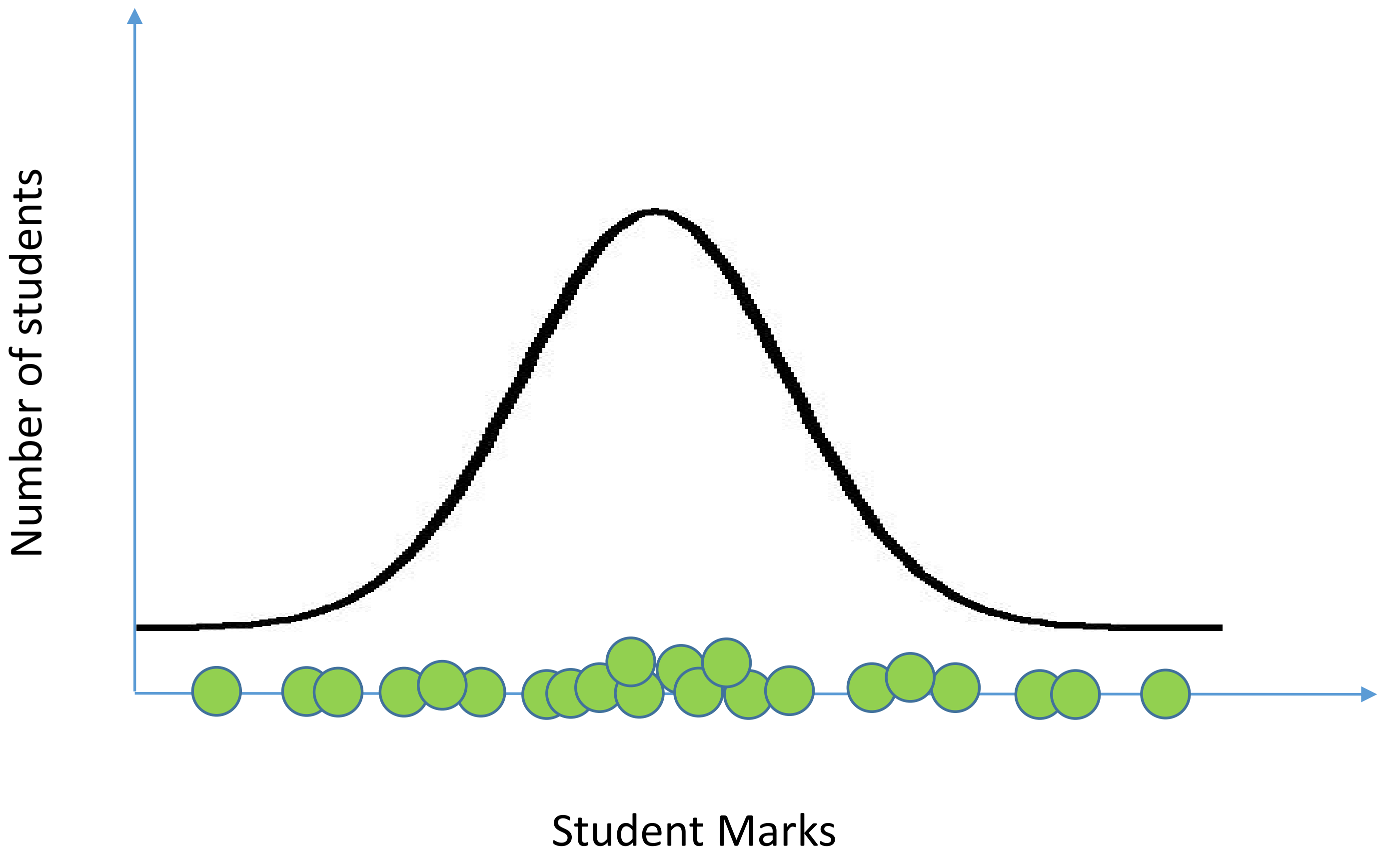


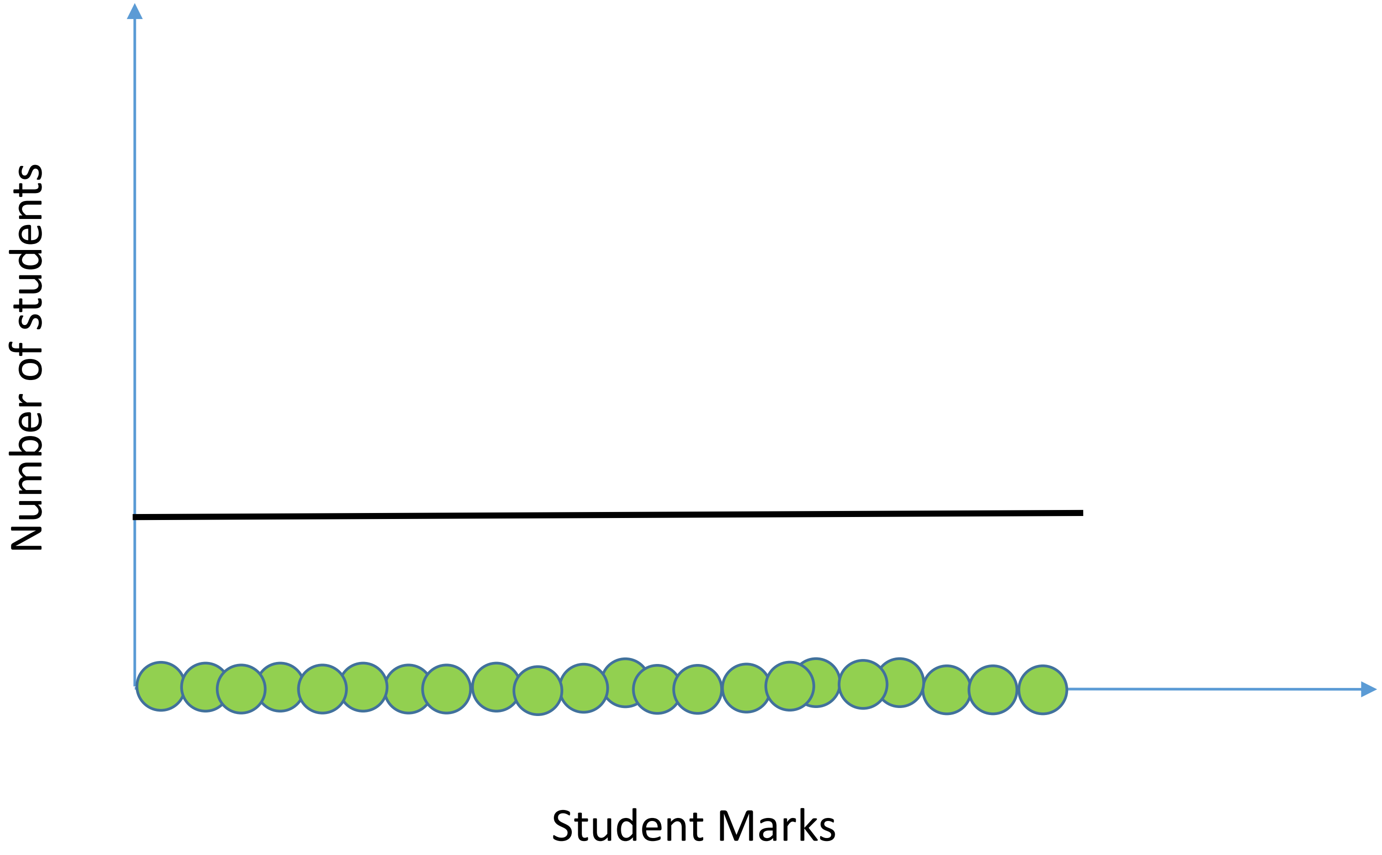
Note: For the distribution to be valid, the assigned probabilities must follow the three axioms of probability!

e.g., assigned probabilities should not be negative.

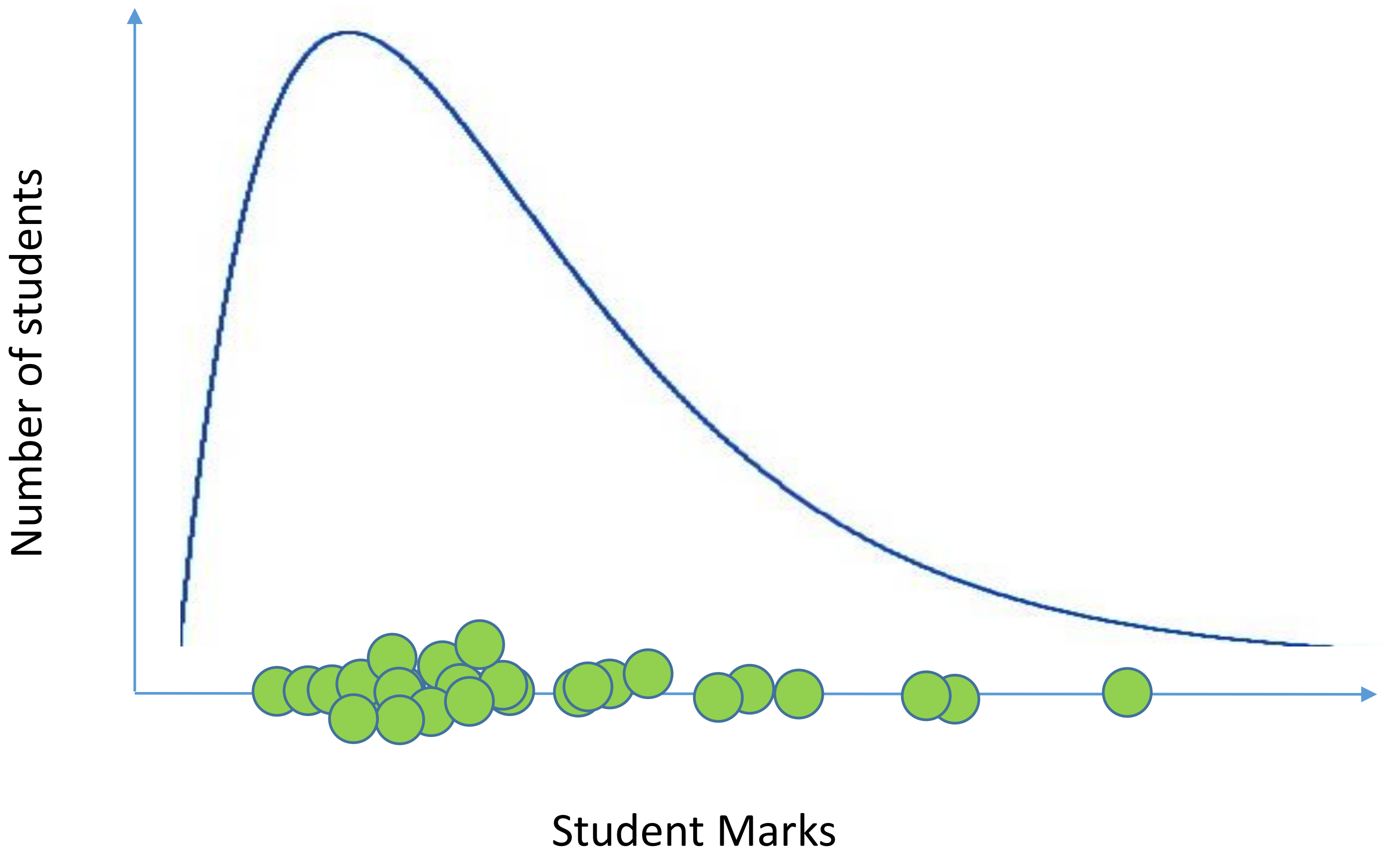
Probability distribution – how to decide the “shape”?



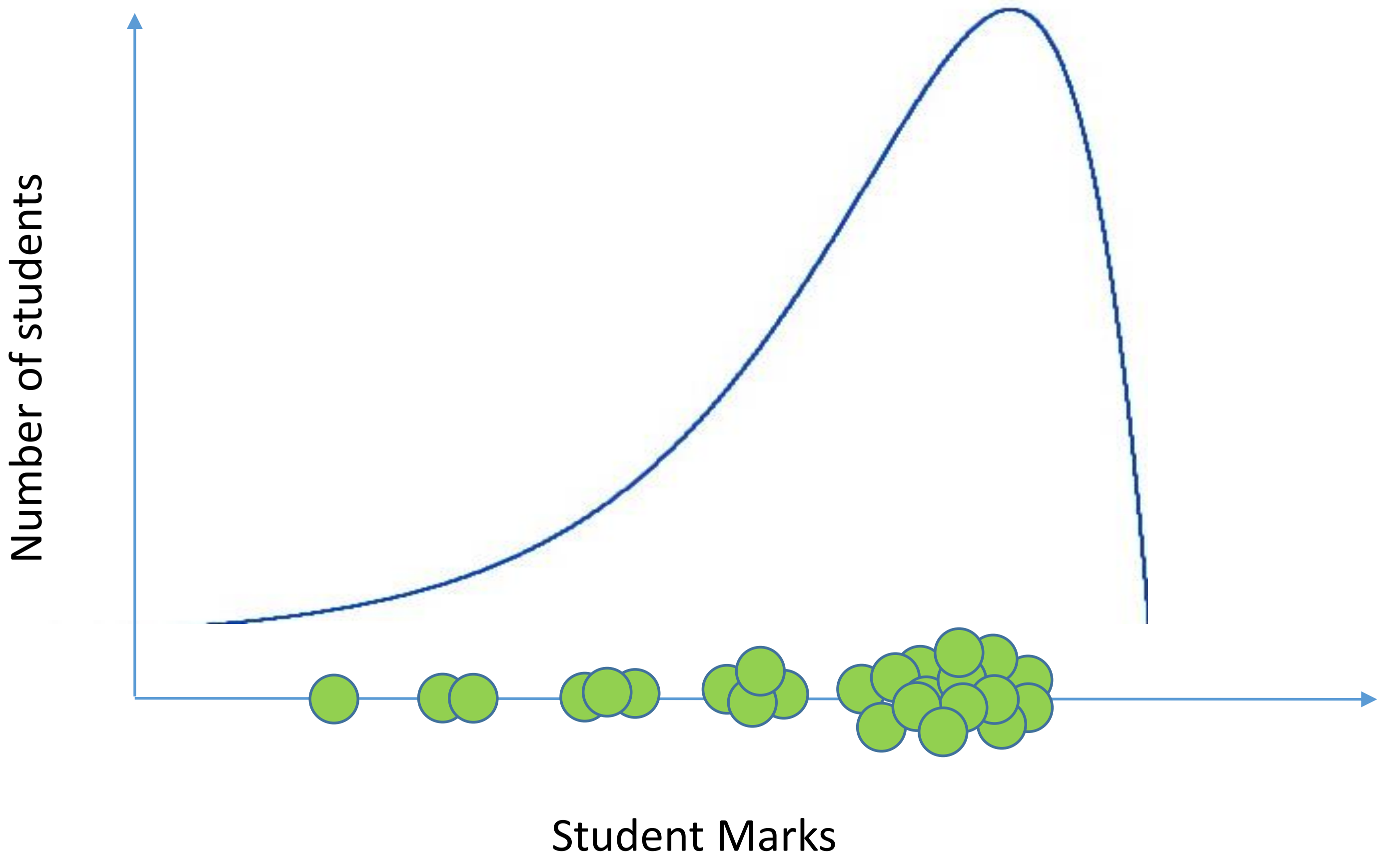
Probability distribution – how to decide the "shape"?



Probability distribution – how to decide the "shape"?



Probability distribution – how to decide the "shape"?



Our friends in an uncertain world!

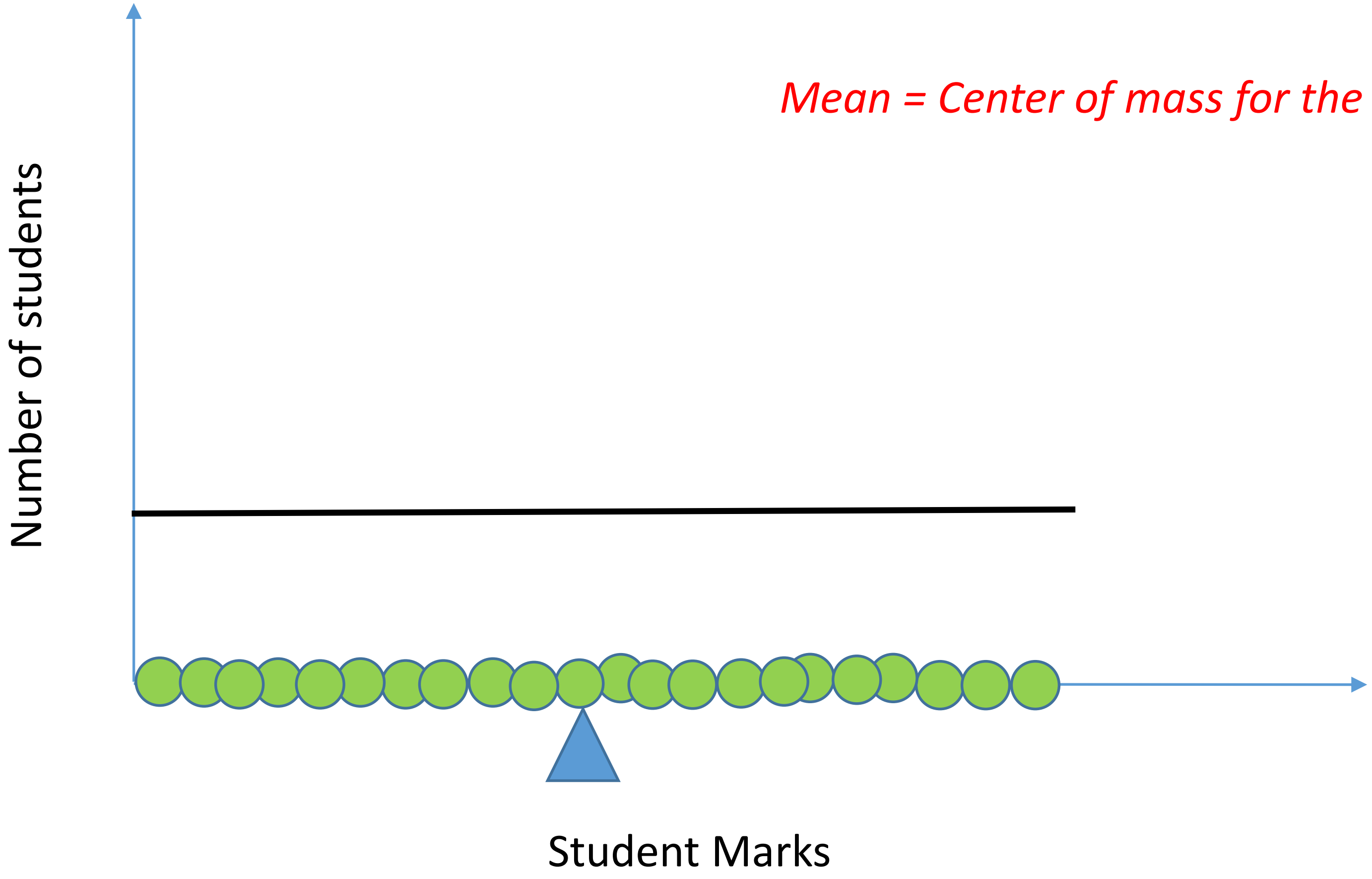
- Random or uncertain events may still have some underlying characteristics that are "fixed"
- Some of these can be

Probability distribution ← If this does not change over time then the random process is **strictly stationary**

↓

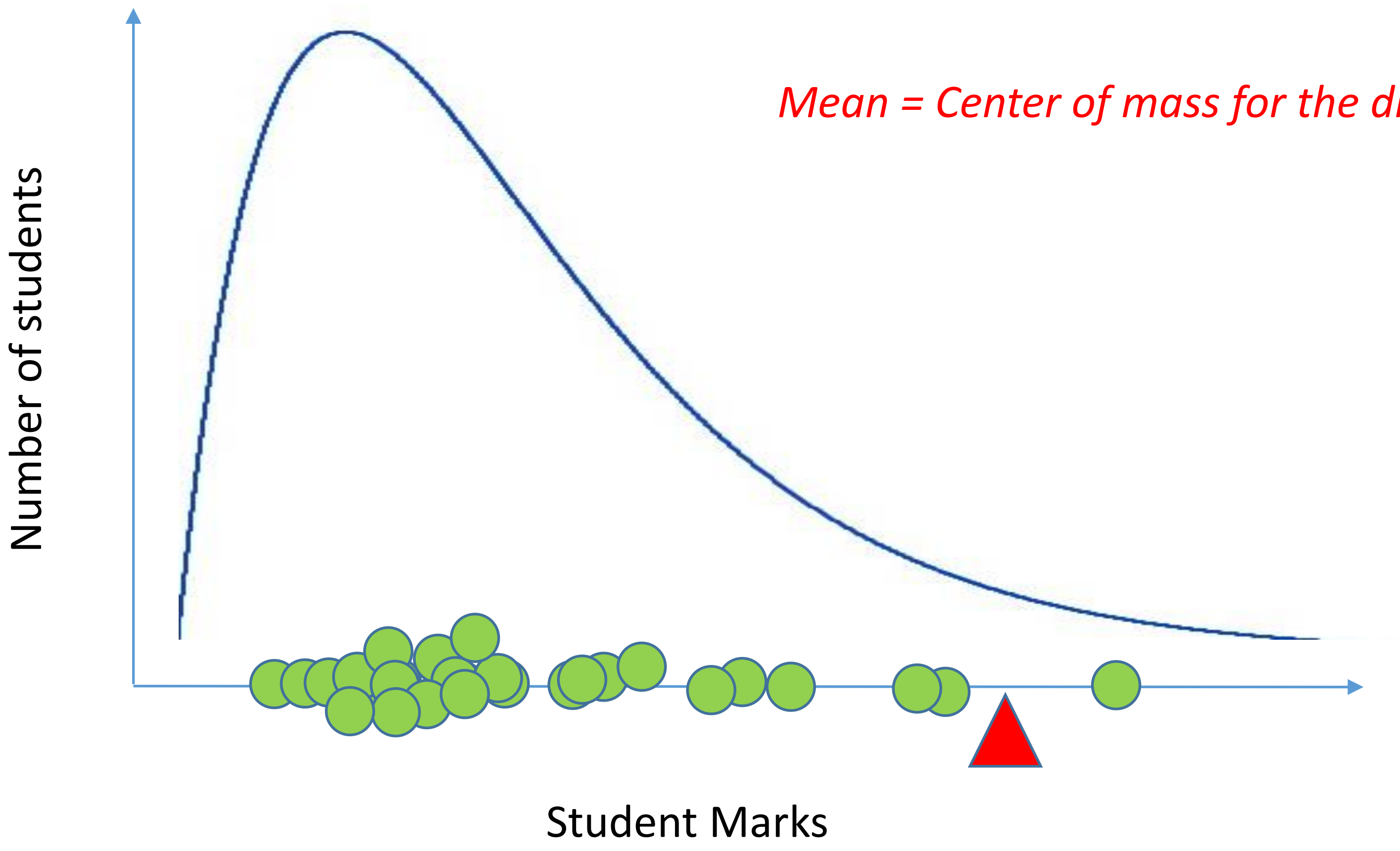
Mean, Variance, Covariance ← If these do not "vary" over time then the random process is **weakly stationary**

Many names: Mean/Average/Expectation



Mean = Center of mass for the distribution (balancing point)

Many names: Mean/Average/Expectation



Mean = Center of mass for the distribution (balancing point)

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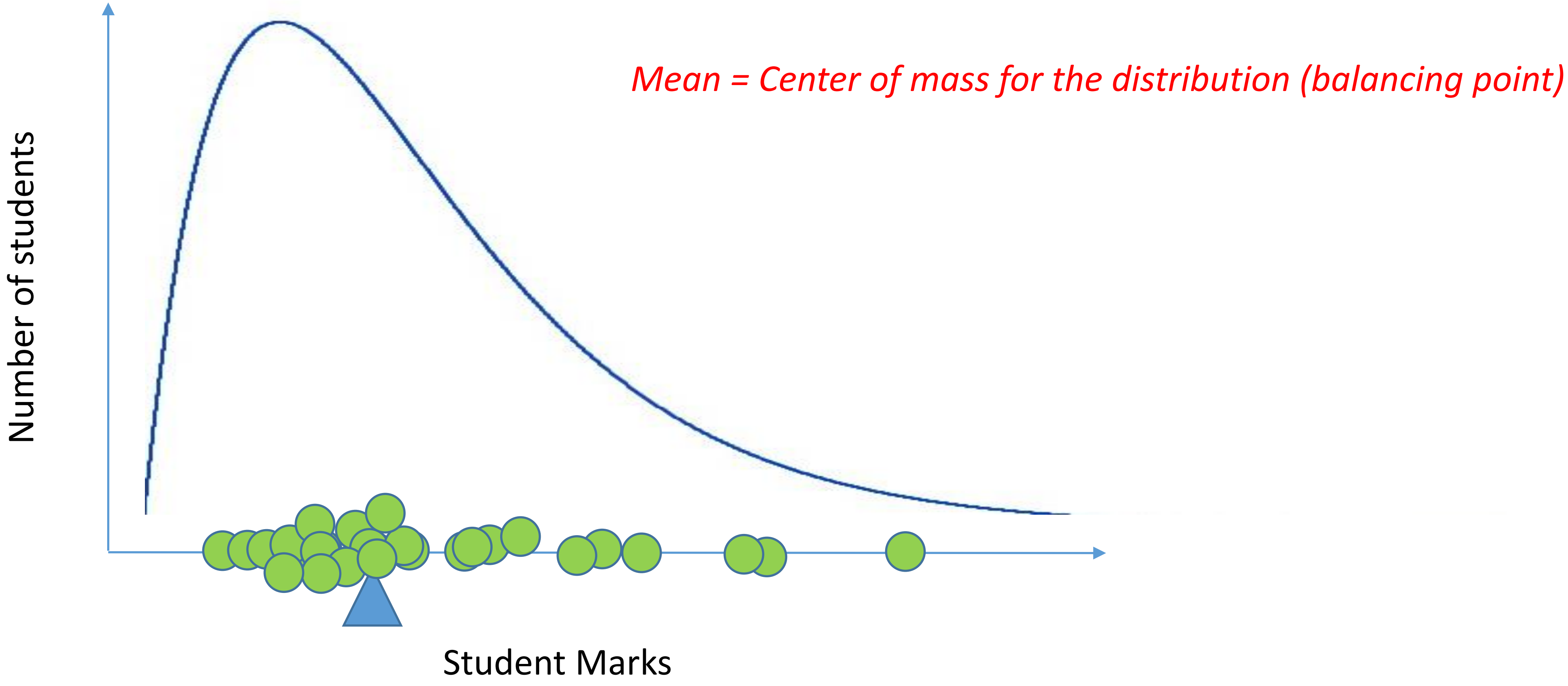
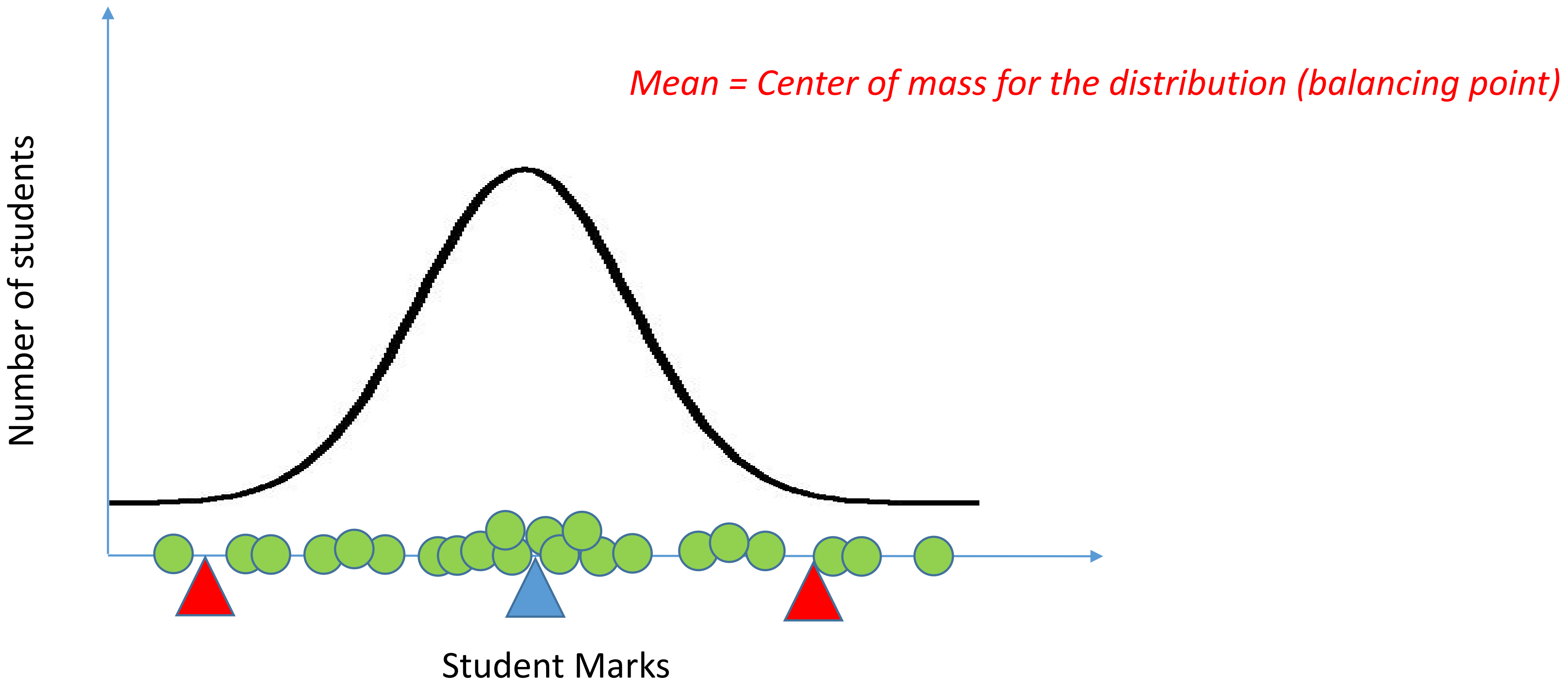


Many names: Mean/Average/Expectation



How do we write the “mean” mathematically?

Discrete case

$$\mu_X = \mathbb{E}[X] = \sum_x x p_X(x)$$

$$\mathbb{E}[g(X)] = \sum_x g(x) p_X(x)$$

P(X = x)

Continuous case

$$\mu_X = \mathbb{E}[X] = \int_{-\infty}^{\infty} x f_X(x)\, dx$$

$$\mathbb{E}[g(X)] = \int_{-\infty}^{\infty} g(x) f_X(x)\, dx$$

Probability density function

$$\mu_X(t) = \mathbb{E}[X(t)]$$

Mean of a random process

Example

- Rolling a die
- $g[x] = 2x$
- Rolling a die five times

What are some of the important properties of the "mean"?

$$\mathbb{E}\left[\sum_{i=1}^{n} X_i\right] = \sum_{i=1}^{n} \mathbb{E}[X_i])$$

$$\mathbb{E}[XY] = \mathbb{E}[X]\mathbb{E}[Y]$$

Only if X and Y are independent

↑

More on this later

Our friends in an uncertain world!

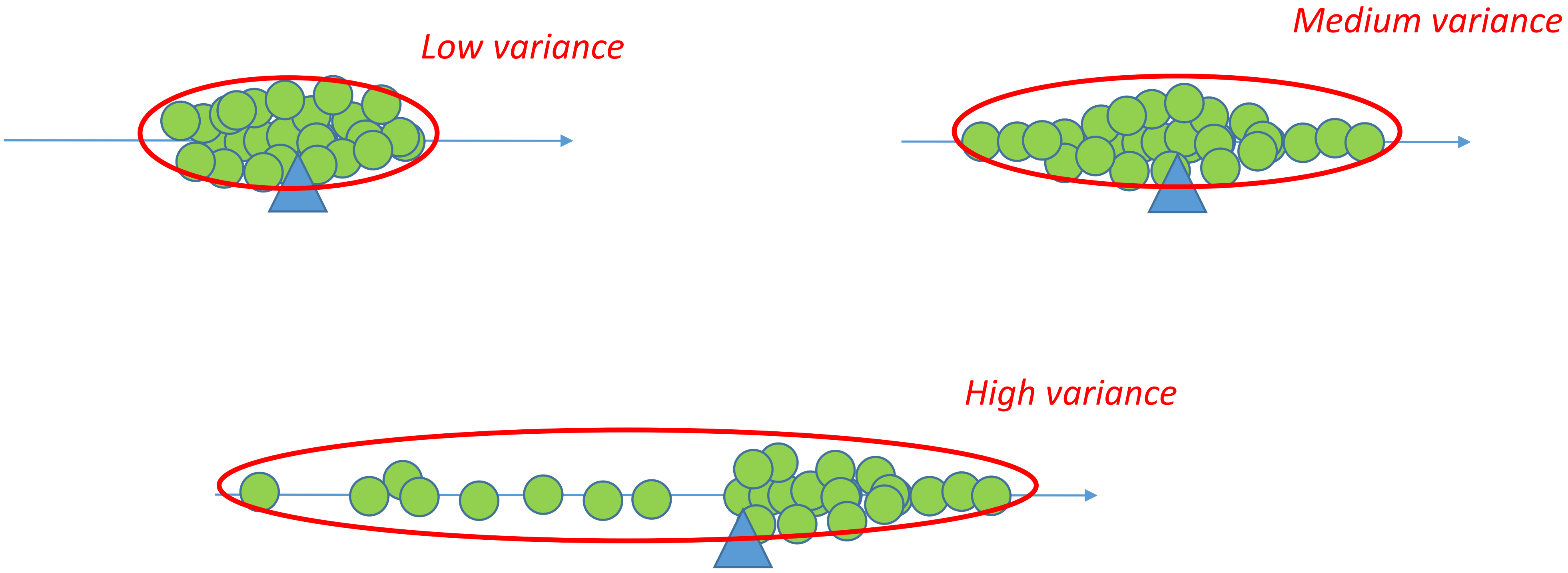
- Random or uncertain events may still have some underlying characteristics that are "fixed"
- Some of these can be

Probability distribution ← If this does not change over time then the random process is **strictly stationary**

↓

Mean, Variance, Covariance ← If these do not "vary" over time then the random process is **weakly stationary**

Variance = degree of spread (how much variation is there in the data/outcomes?)



How do we write "variance" mathematically?

Discrete case

$$\text{var}[X] = \mathbb{E}(X - \mu_x)^2$$

$$= \sum_x (x - \mu_X)^2 \, p_X(x)$$

Continuous case

$$\text{var}[X] = \mathbb{E}(X - \mu_x)^2$$

$$= \int_{-\infty}^{\infty} (x - \mu_X)^2 f_X(x) \, \mathrm{d}x$$

$$\text{var}[X] = \mathbb{E}[X^2] - \mu_X^2$$

Example

- Rolling a die

Our friends in an uncertain world!

- Random or uncertain events may still have some underlying characteristics that are "fixed"
- Some of these can be

Probability distribution ← If this does not change over time then the random process is **strictly stationary**

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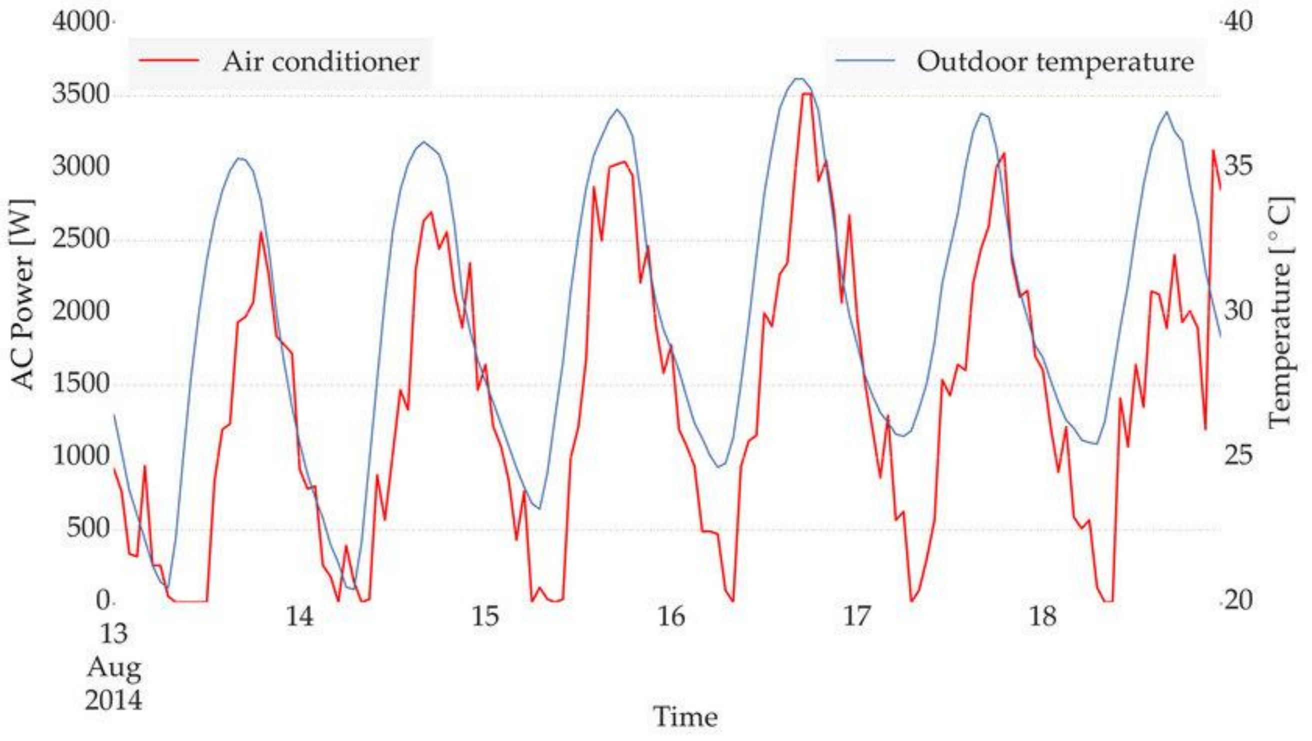
Mean, Variance, Covariance ← If these do not "vary" over time then the random process is **weakly stationary**

Covariance = degree of linear relationship between data/outcomes

Can knowledge of one random process help us say something about the value of another?

Yes – if their covariance is high!



Covariance = degree of linear relationship between data/outcomes

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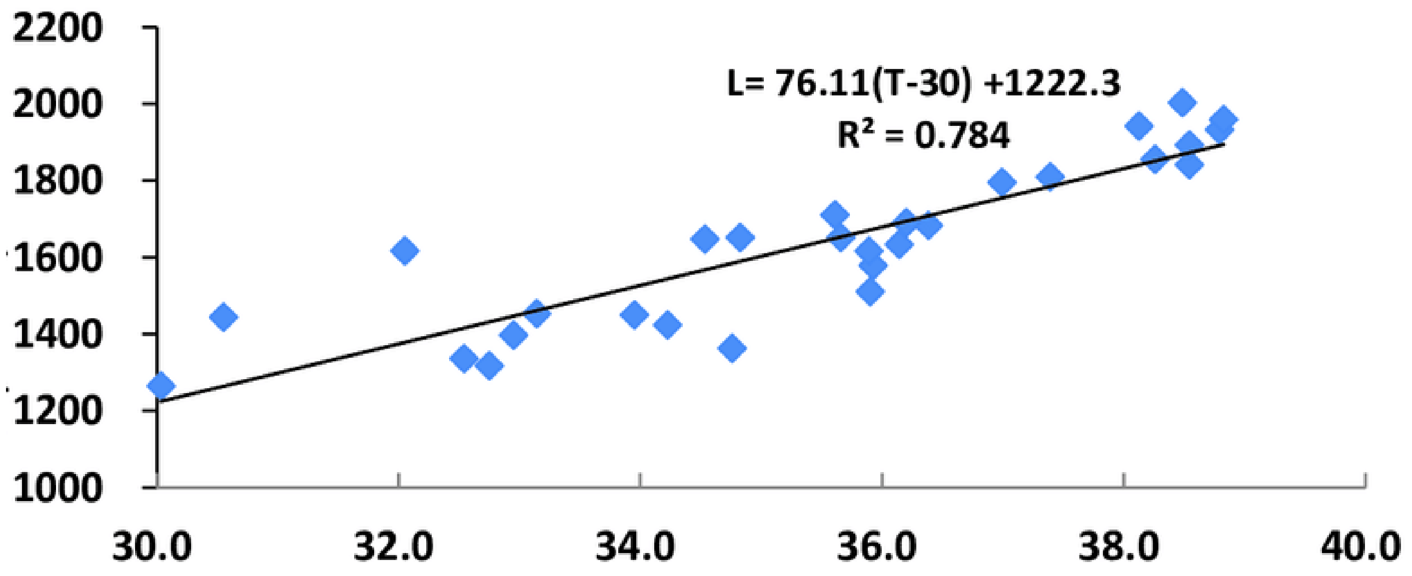
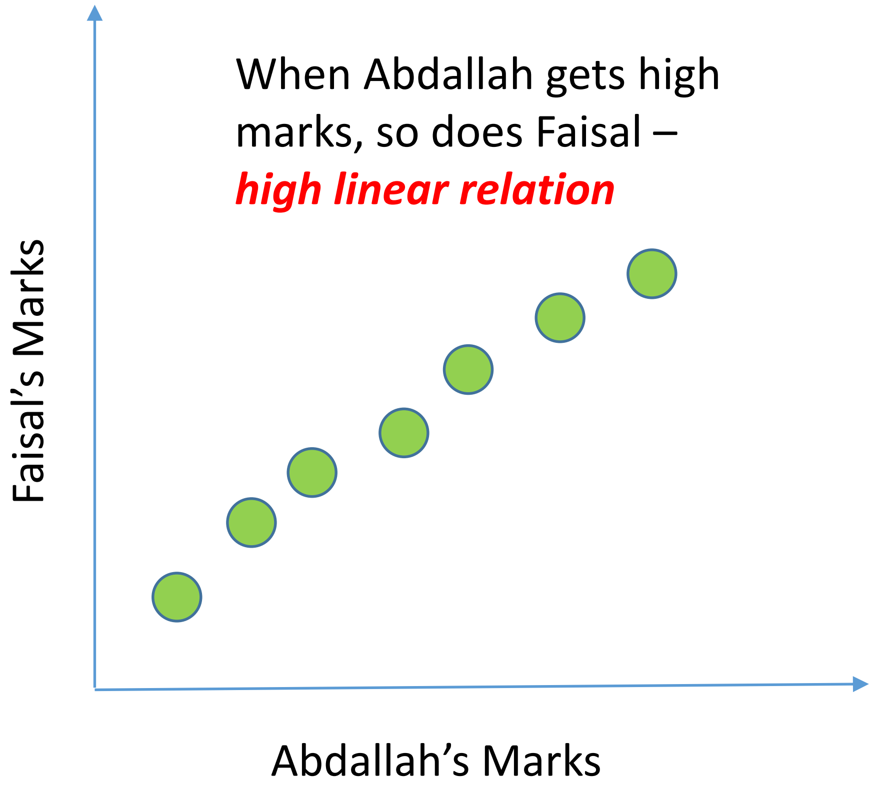


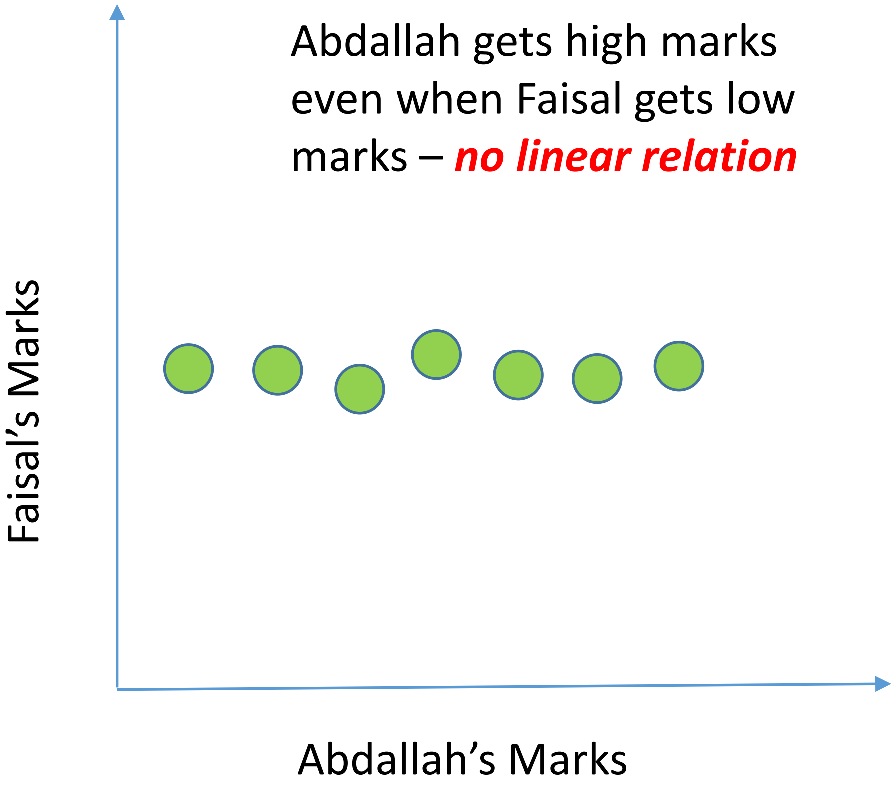
One way of checking linear relationship ("covariance") is to plot the two variables against each other.

Covariance = degree of linear relationship between data/outcomes

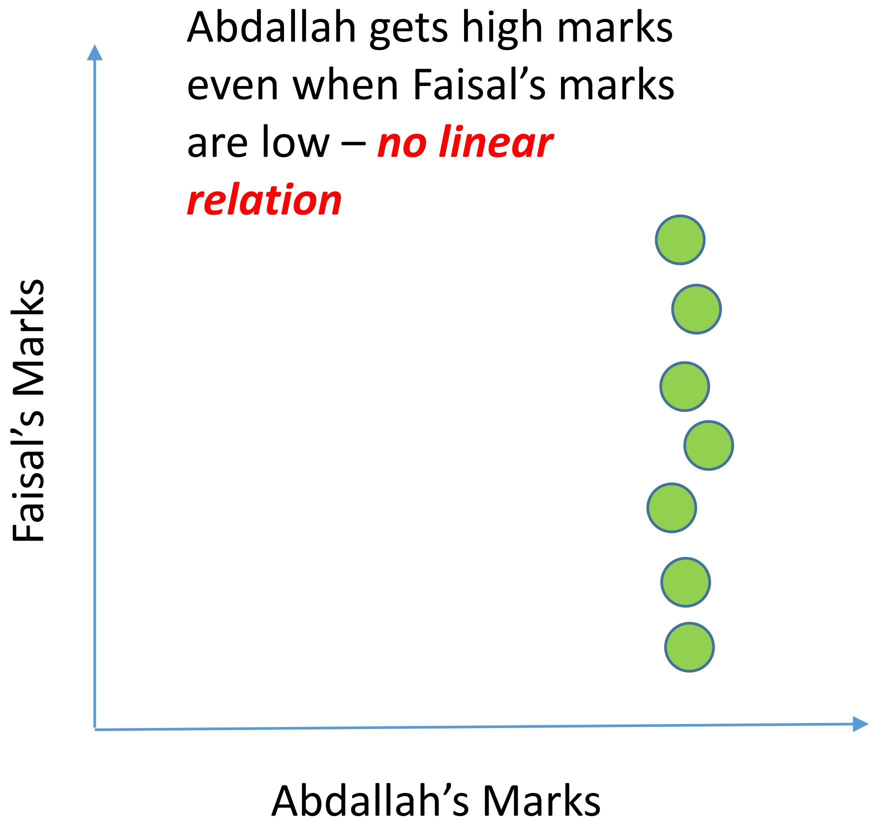
	Abdallah's Marks	Faisal's Marks
Quiz 1	4	4
Quiz 2	8	7
Quiz 3	2	1
Quiz 4	4	5
Quiz 5	1	0
Quiz 6	9	10

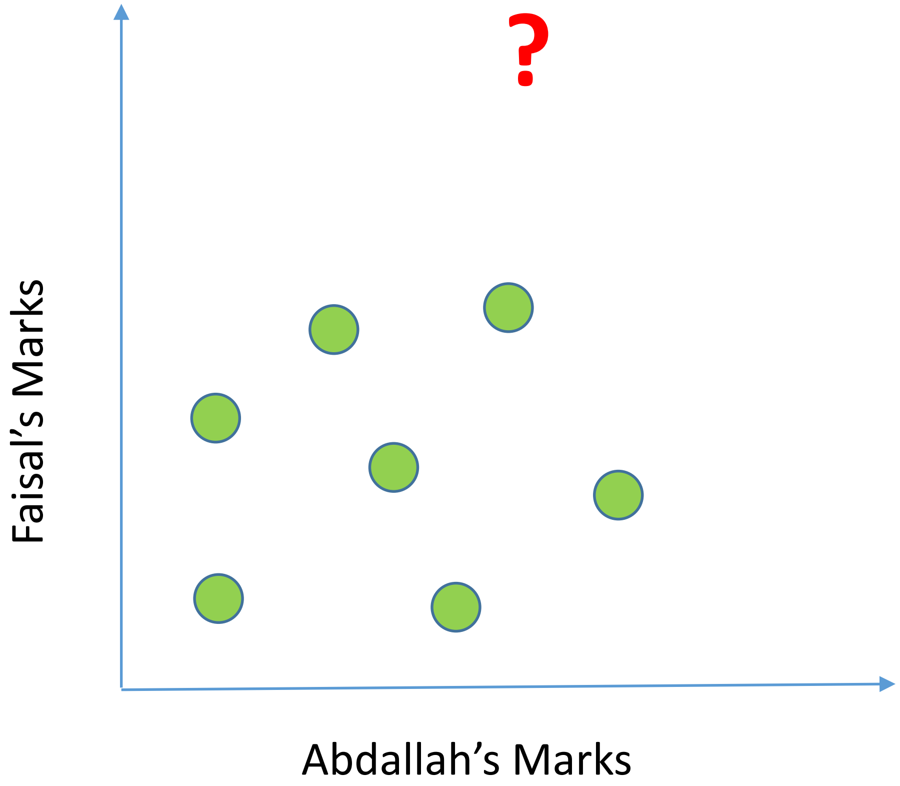
Covariance = degree of linear relationship between data/outcomes





Covariance = degree of linear relationship between data/outcomes





Covariance = degree of linear relationship between data/outcomes

- Covariance helps us understand if two random processes are statistically related or not
  - **Statistically independent processes have zero covariance**
- We usually normalize covariance so that it lies between -1 and 1
  - Normalized covariance is called "**Correlation**"
- Noise in communications is mostly assumed to be independent of the message signal (i.e., we often assume zero correlation between signal and noise)

How do we write "covariance" mathematically?

Covariance of two random variables X and Y

$$C(X,Y) = \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])]$$
$$= \mathbb{E}[XY] - \mu_X \mu_Y$$

Independent random variables have zero covariance

$$C(X,Y) = 0$$

For a random process we use the ***autocovariance function***

$$C_{XX}(t_1, t_2) = \mathbb{E}[(X(t_1) - \mu_{X_{t_1}})(X(t_2) - \mu_{X_{t_2}})]$$

Covariance between two samples of random process X(t)

Example

- Rolling two dice at the same time
- Rolling a die five times

Our friends in an uncertain world!

- Random or uncertain events may still have some underlying characteristics that are "fixed"
- Some of these can be

Probability distribution ← If this does not change over time then the random process is **strictly stationary**

↓

Mean, Variance, Covariance ← If these do not "vary" over time then the random process is **weakly stationary**

Mean and Autocovariance of a Weakly Stationary Process

The mean of a weakly stationary process does not change with time

$$\mu_X(t) = \mu_X \qquad \text{for all } t$$

The autocovariance function of a weakly stationary process depends only on the time difference between the samples and NOT on their actual values

$$C_{XX}(t_2 - t_1) \qquad \text{for all } t_1 \text{ and } t_2$$

or

$$C_{XX}(\tau) \qquad \tau = t_2 - t_1$$

Example

- Some functions that satisfy second condition

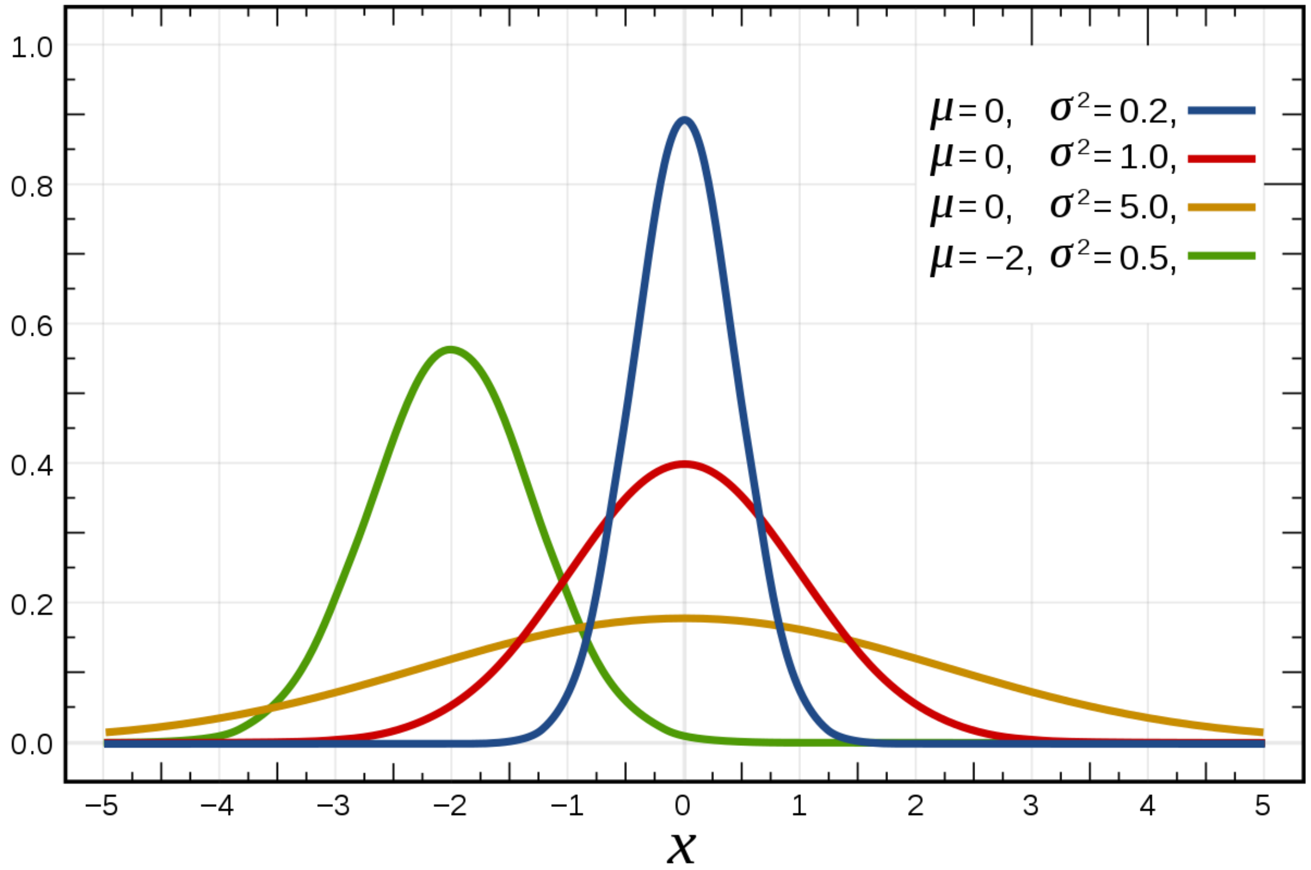
Back to Gaussian Distribution …

- With the intuitive knowledge of distribution, mean, variance, and covariance we are now ready to take a closer look at the very special, common, and useful distribution – the Gaussian Distribution.

mean

$$f_X(x) = \frac{1}{\sqrt{2\pi}\,\sigma}\exp\left[-\frac{(x-\mu)^2}{2\sigma^2}\right]$$

variance



Some Practice Problems …

Consider a discrete r.v. X whose pmf is given by

$$p_X(x) = \begin{cases} \frac{1}{3} & x = -1, 0, 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Plot $p_X(x)$ and find the mean and variance of X.

(b) Repeat (a) if the pmf is given by

$$p_X(x) = \begin{cases} \frac{1}{3} & x = -2, 0, 2 \\ 0 & \text{otherwise} \end{cases}$$

Some Practice Problems …

(a)

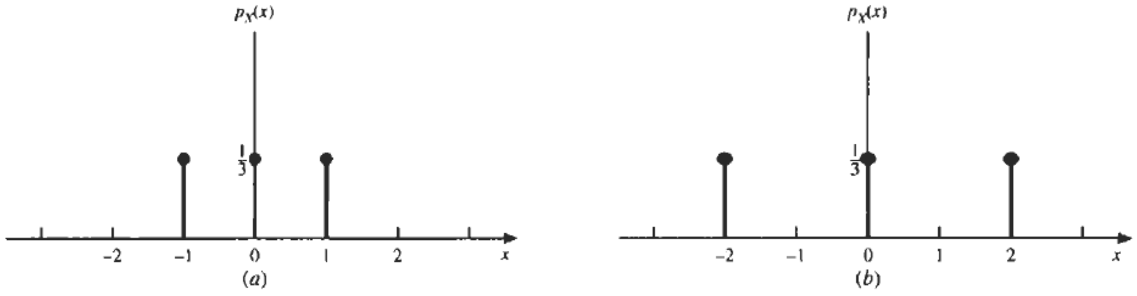
$$\mu_X = E(X) = \tfrac{1}{3}(-1 + 0 + 1) = 0$$

$$\sigma_X^2 = \mathrm{Var}(X) = E[(X - \mu_X)^2] = E(X^2) = \tfrac{1}{3}[(-1)^2 + (0)^2 + (1)^2] = \tfrac{2}{3}$$

(b)

$$\mu_X = E(X) = \tfrac{1}{3}(-2 + 0 + 2) = 0$$

$$\sigma_X{}^2 = \mathrm{Var}(X) = \tfrac{1}{3}[(-2)^2 + (0)^2 + (2)^2] = \tfrac{8}{3}$$



Some Practice Problems …

A continuous-time random process is given as $X(t) = A + Bt$, where A and B are independent random variables uniformly distributed in the interval $[-1, 1]$. Find:

(a) The mean function $m_x(t)$

(b) The covariance function $r_x(s, t)$

(c) Is $X(t)$ wide-sense stationary (WSS)?

Some Practice Problems …

(a) Since A and B are uniformly distributed over the interval $[-1,1]$, it is easy to see that they are both zero-mean, i.e., $E[A] = E[B] = 0$. With that, we get

$$m_x(t) = E[A + Bt] = E[A] + E[B]t = 0$$

(b)

$$\begin{aligned} r_x(s,t) &= C[X(s), X(t)] = C[A + Bs, A + Bt] \\ &= V[A] + tC[A,B] + sC[A,B] + stV[B] \end{aligned}$$

Since A and B are independent, we have $C[A,B] = 0$. Furthermore

$$V[A] = V[B] = \int_{-1}^{1} \frac{x^2}{2} dx = \frac{1}{3}$$

Therefore

$$r_x(s,t) = \frac{1}{3} + \frac{1}{3} st$$

(c) The process is not WSS as the covariance function does not depend only on the lag $s - t$.

Questions?? Thoughts??

EE 322
Digital Communications

with

Dr. Naveed R. Butt

@

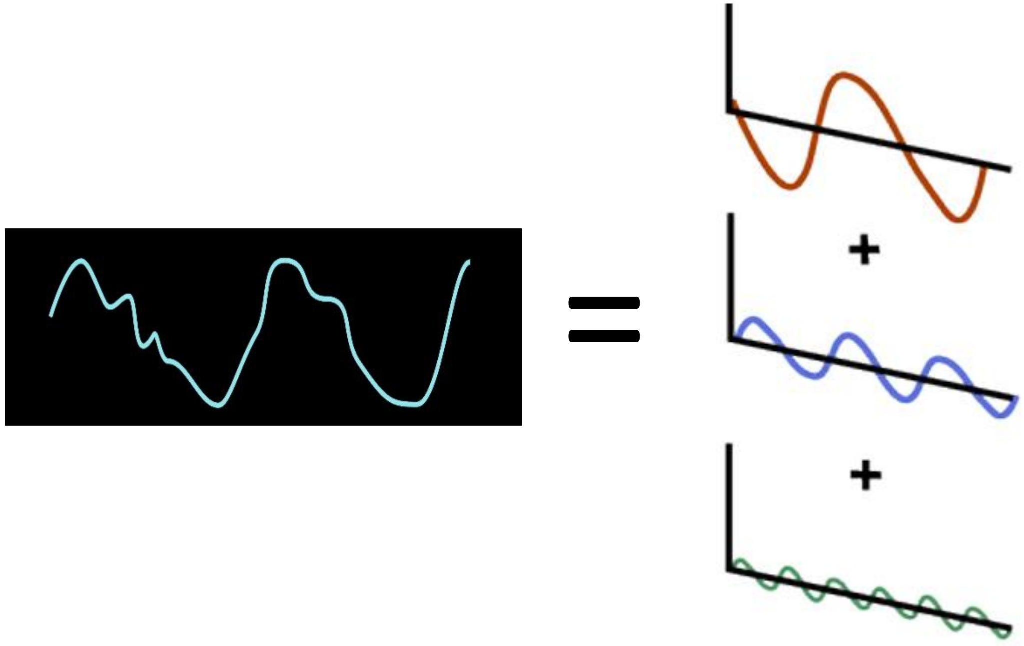
Jouf University

Some familiar terms …

- Probability
- Gaussian Noise
- Fourier Transform ⇦
- Power Spectrum
- White Noise

We saw in a previous lecture that …

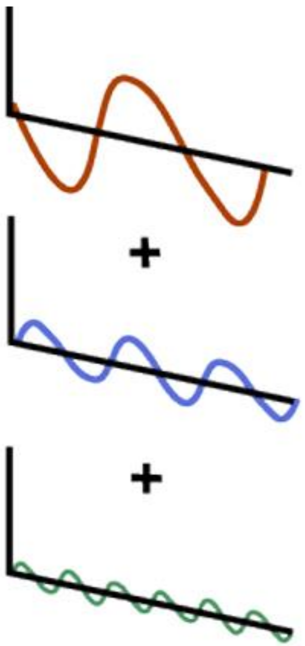
- In Fourier Transform, we want to look at signals in terms of a fixed set of ingredients
  - Ingredients : Sinusoids of different frequencies



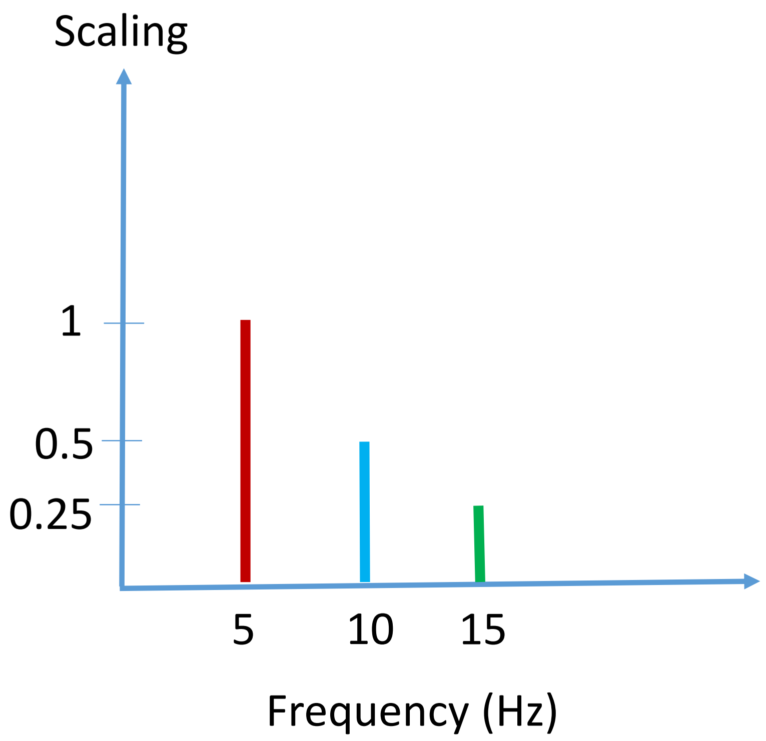
Ingredient (sinusoid frequency)	Amount (scaling)	Process
f_1	1	**Add all**
f_2	0.5	
f_3	0.25	

Fourier Transform

- How is this shown after Fourier transform?

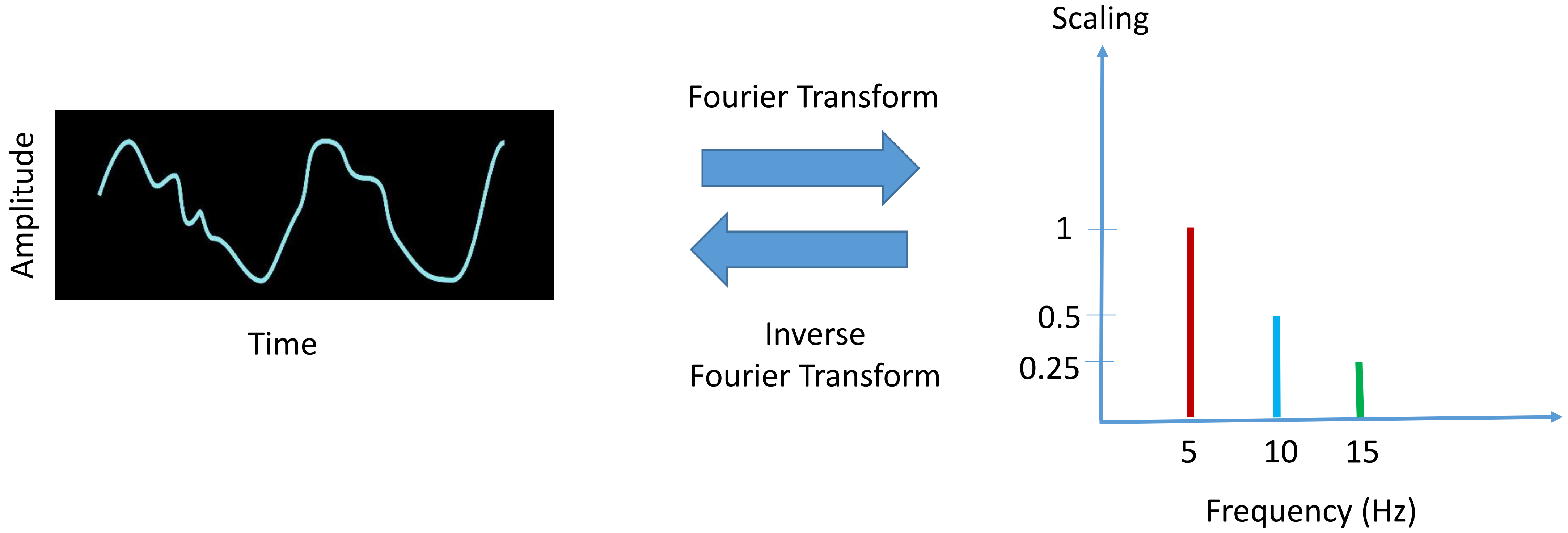


Ingredient (sinusoid frequency)	Amount (scaling)	Process
5 Hz	1	**Add all**
10 Hz	0.5	
15 Hz	0.25	



Fourier Transform

- How is this shown after Fourier transform?



How do we write the Fourier Transform mathematically?

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi ft)\,dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi ft)\,df$$

← *Shows g(t) written as a "sum" (integral) of complex sinusoids*

$$G(f) = \mathbf{F}[g(t)]$$

$$g(t) = \mathbf{F}^{-1}[G(f)]$$

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\ \cos(\omega t) &= \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \\ \sin(\omega t) &= \tfrac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \end{aligned}$$

How do we write the Fourier Transform mathematically?

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

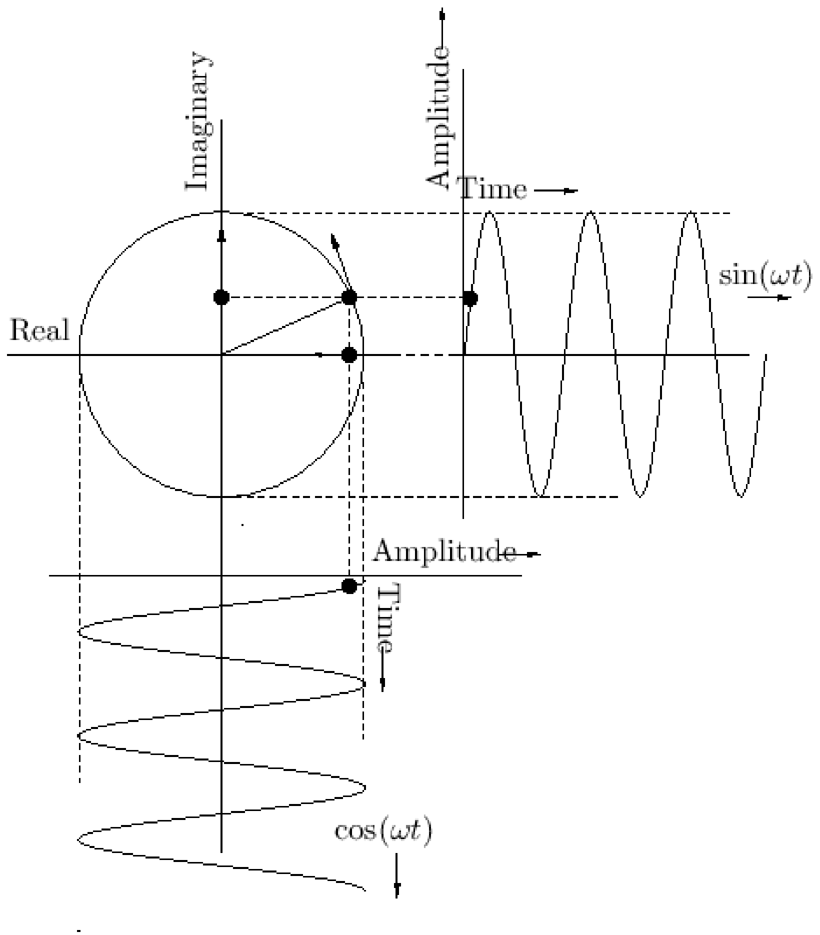
Fourier Transform of discrete time process x[n]

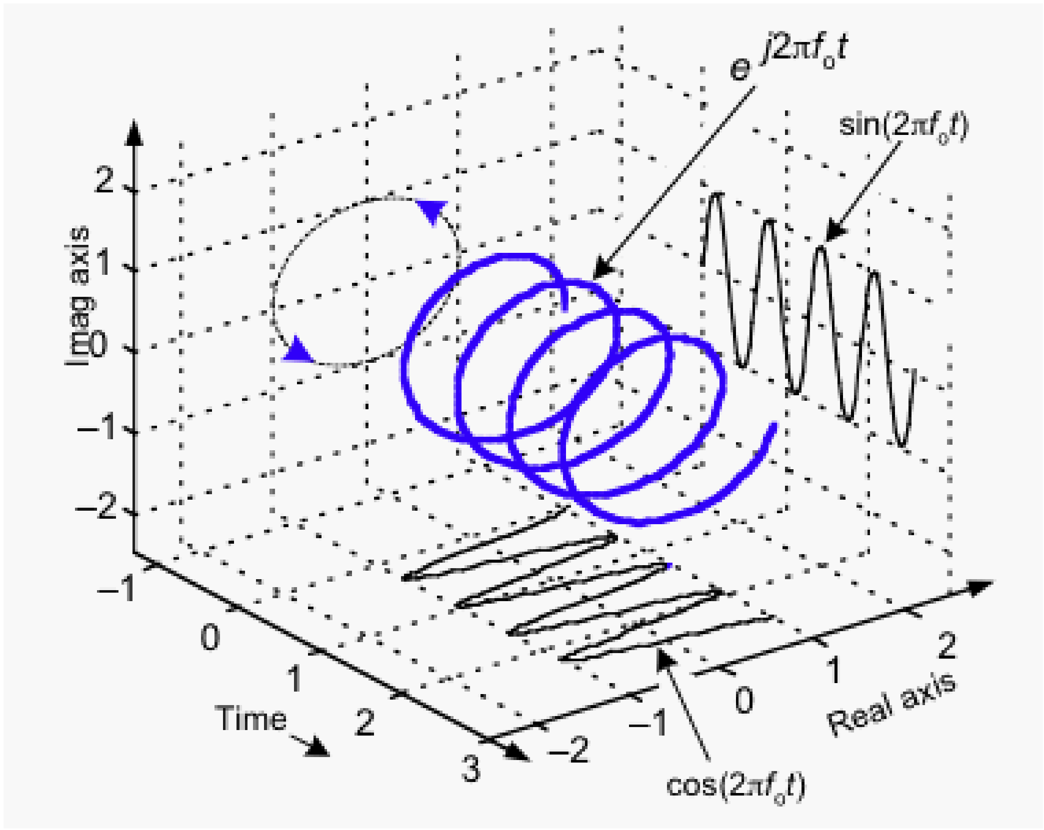
Why complex sinusoids?

- Mathematically easier to work with (e.g., product)

Visualizing Complex Sinusoids

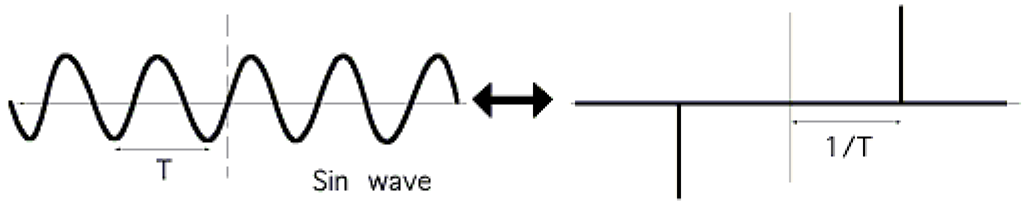
$$e^{j\omega t} = \cos(\omega t) + j\sin(\omega t)$$

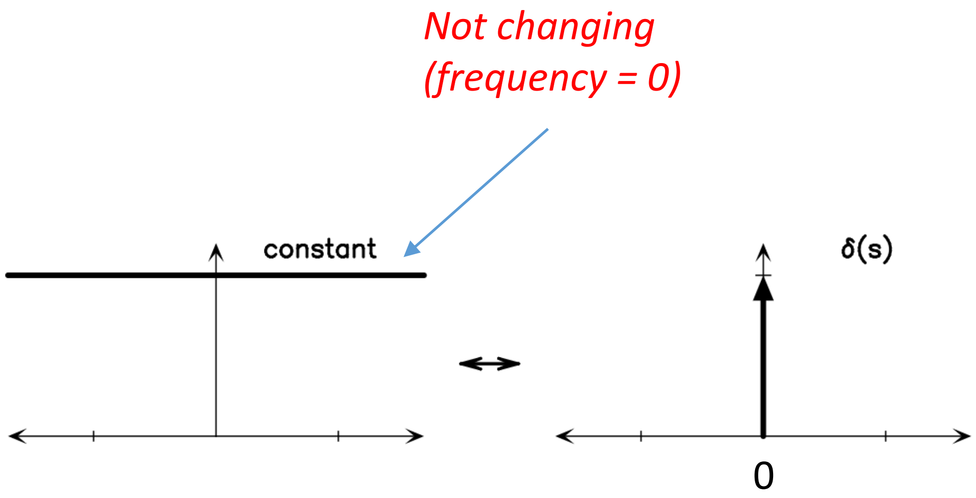




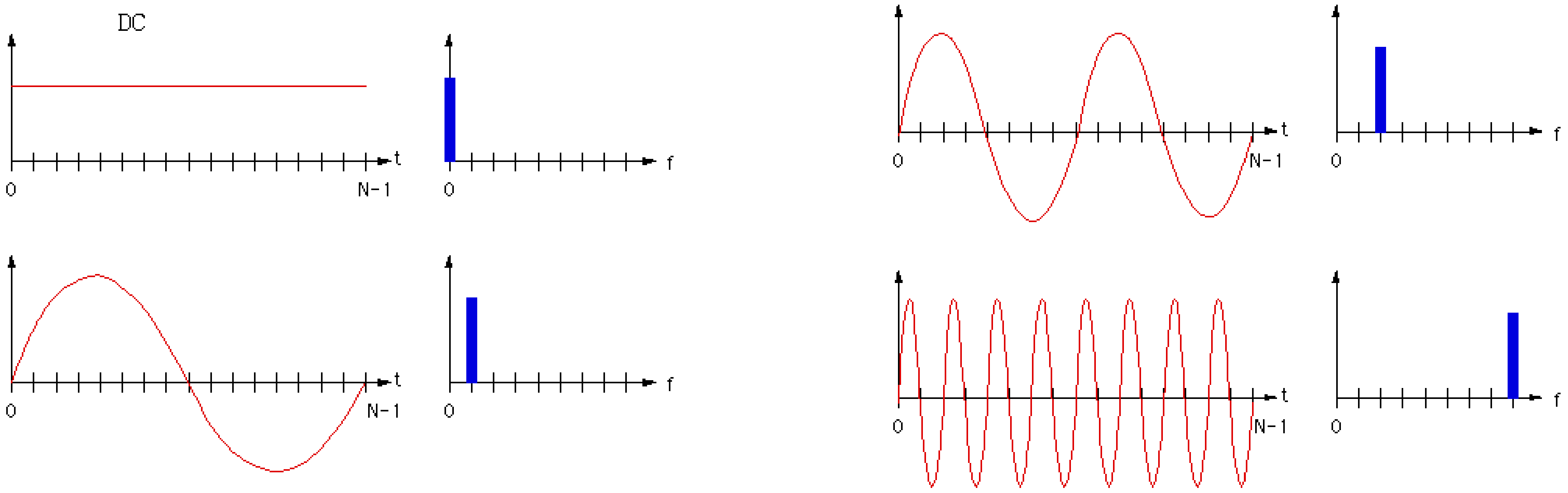
Some Fourier Transforms (Visual)

$$\sin(\omega t) \;=\; \frac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right)$$



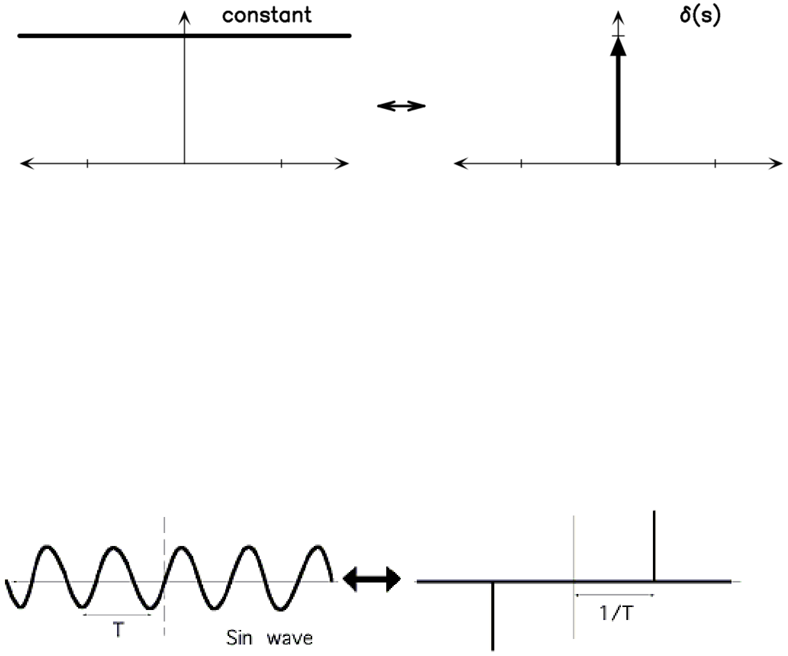


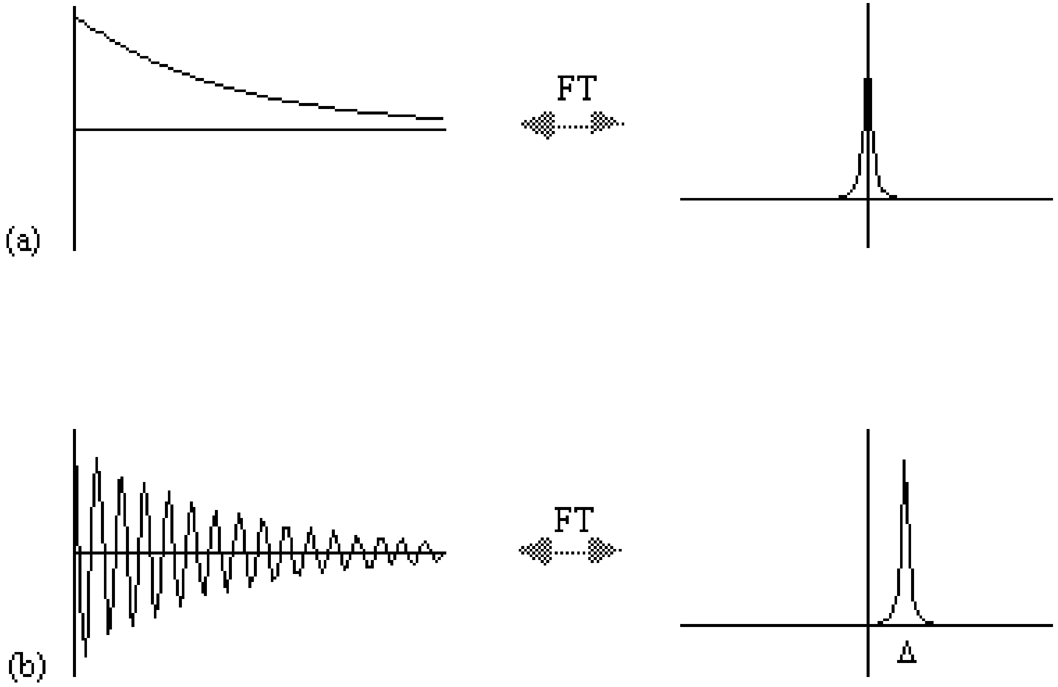
Some Fourier Transforms (Visual)



As frequencies increase, the FT peaks move outwards

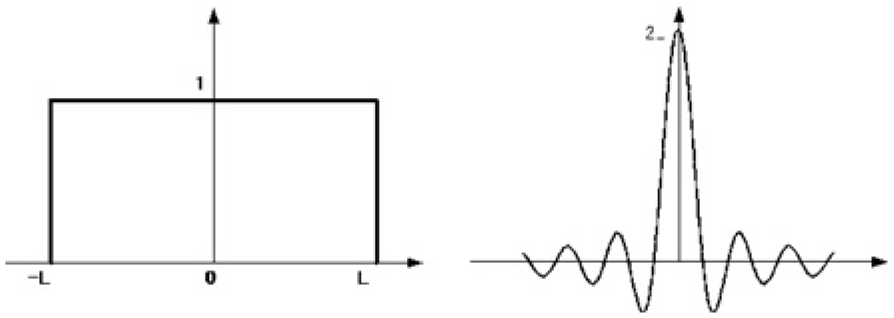
Some Fourier Transforms (Visual)



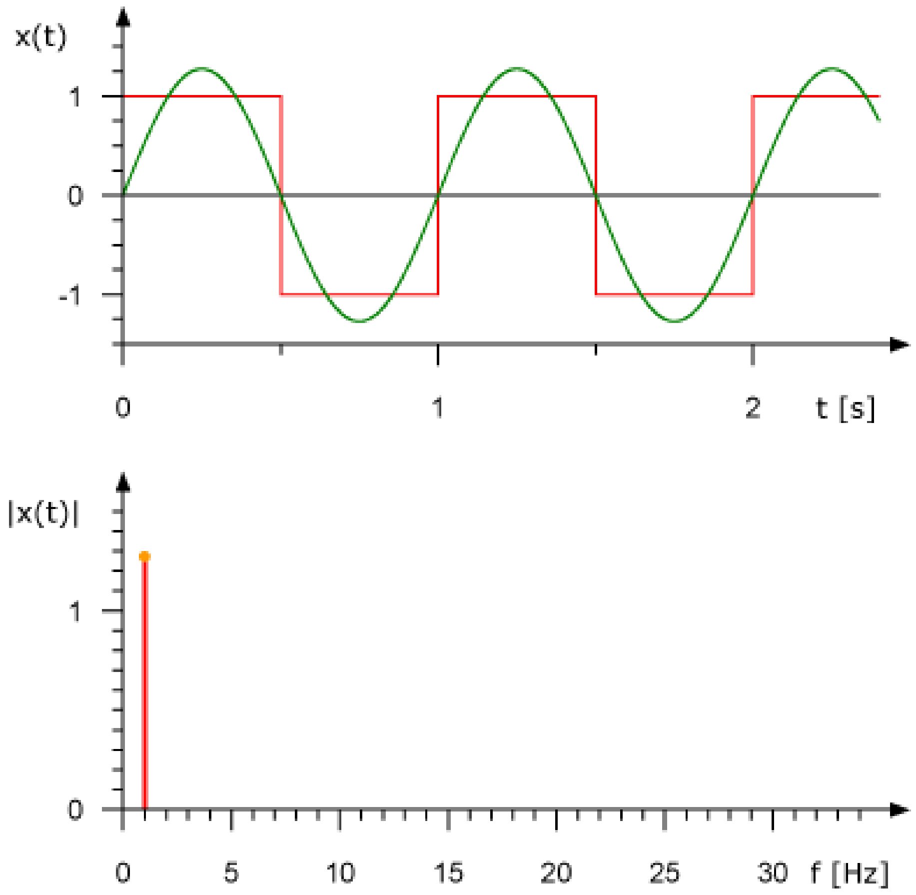


"Damping" causes "spread"

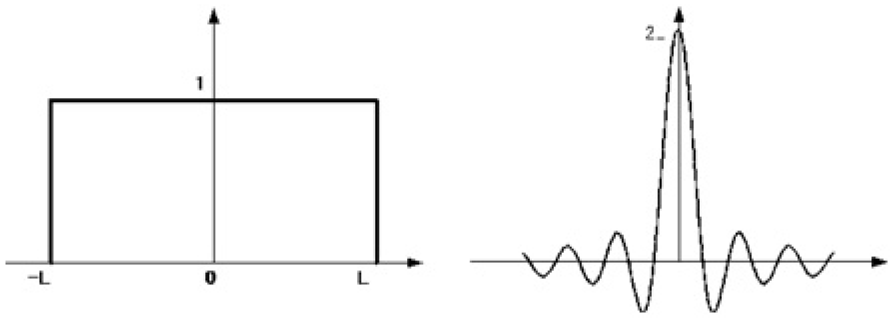
Some Fourier Transforms (Visual)



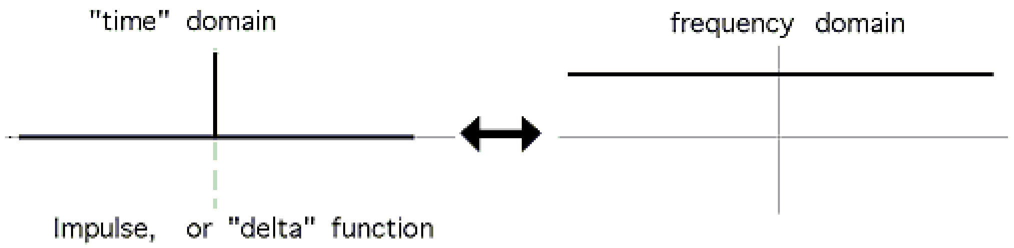
Sharp changes (edges)
require a lot of frequencies



Some Fourier Transforms (Visual)



Sharp changes (edges)
require a lot of frequencies



… and an extremely sharp change
(delta) requires ALL the frequencies!!

Some familiar terms …

- Probability
- Gaussian Noise
- Fourier Transform
- Power Spectrum
- White Noise

Our friends in an uncertain world!

From previous lecture

- Random or uncertain events may still have some underlying characteristics that are "fixed"
- Some of these can be

Probability distribution ← If this does not change over time then the random process is **strictly stationary**

↓

Mean, Variance, Covariance ← If these do not "vary" over time then the random process is **weakly stationary**

Autocovariance and Autocorrelation of a weakly stationary process

Autocovariance function

$$C_{XX}(\tau) = \mathbb{E}[(X(t+\tau) - \mu_X)(X(t) - \mu_X)]$$

Covariance between two samples of random process X(t)

Only a minor difference between the two. Conceptually very similar.

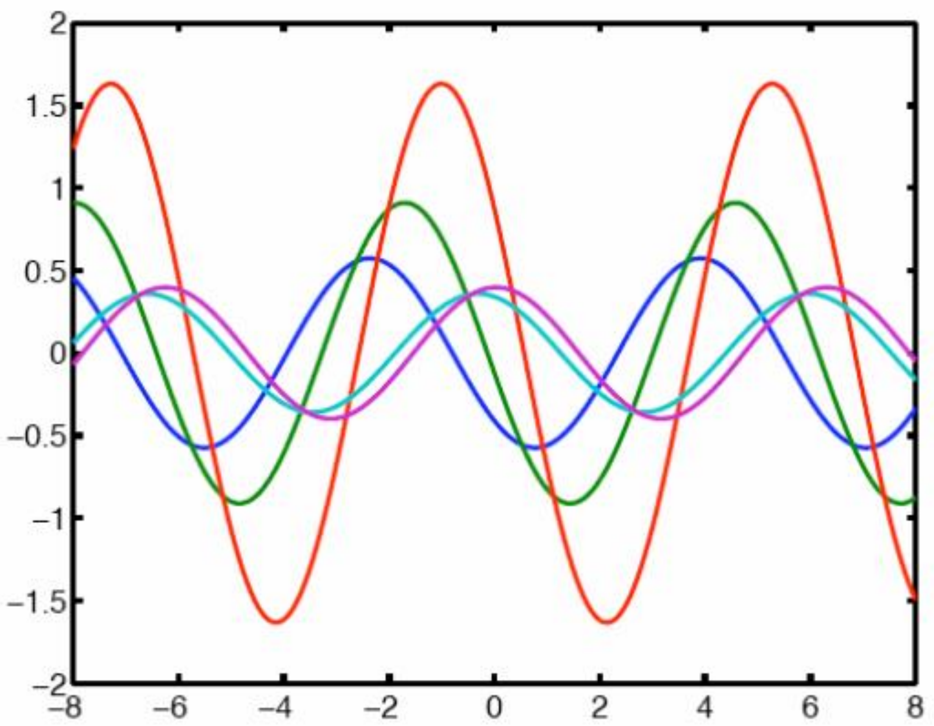
Correlation between two samples of random process X(t)

Autocorrelation function

$$R_{XX}(\tau) = \mathbb{E}[X(t+\tau)X(t)]$$

Frequency content of a random process

- We've seen that
  - Fourier Transform can be used to see the "frequency content" of a signal
- But what if the signal is random?
  - **Problem**: frequency content may change from one realization to another!



$$X(t) = A\cos(\omega t + \phi).$$

A and ϕ random variables

Autocorrelation Function to the rescue!!

- We’ve seen that
  - Fourier Transform can be used to see the “frequency content” of a signal
- But what if the signal is random?
  - **Problem**: frequency content may change from one realization to another!
- Two important observations can help us out
  - 1. The autocorrelation function of a weakly stationary process remains “fixed” between realizations
  - 2. The autocorrelation function contains the same frequencies as the original signal (with “average power” scalings)
- Solution:
  - For a random process it is better to **take the Fourier Transform of the covariance function**

For a random process it is better to take the Fourier Transform of the autocorrelation function!!

- **Power Spectrum** = Fourier Transform of the autocorrelation function
- Mathematically speaking …

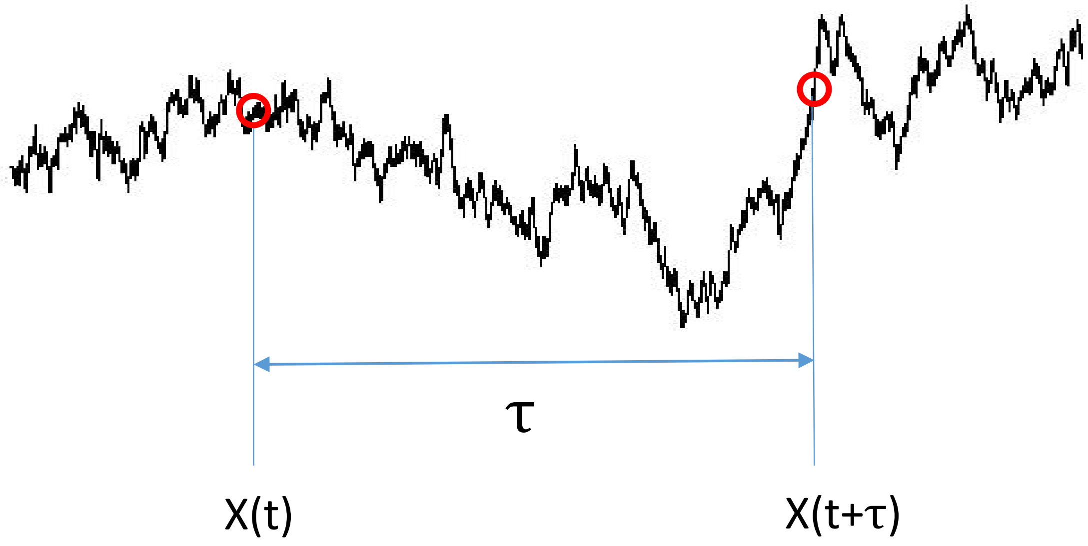
$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\ \exp(-j2\pi f\tau)\ d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)\ \exp(j2\pi f\tau)\ df$$

Calculating the autocorrelation function

- For a weakly stationary process X(t)

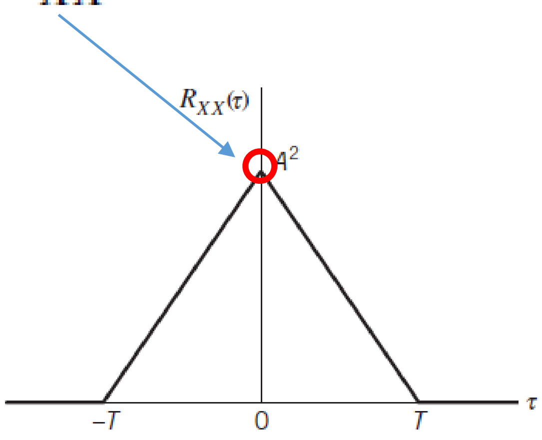
$$R_{XX}(\tau) = \mathbb{E}[X(t+\tau)X(t)]$$



Calculating the autocorrelation function

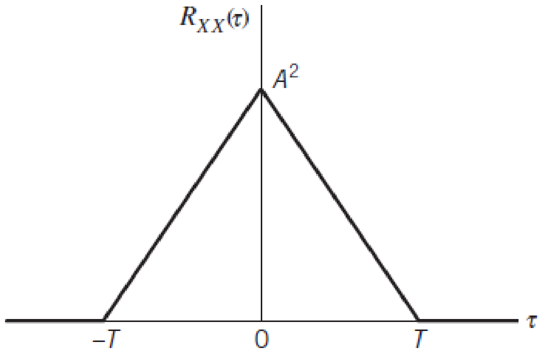
- Three important properties of the autocorrelation function

$$R_{XX}(0) = \mathbb{E}[X^2(t)]$$



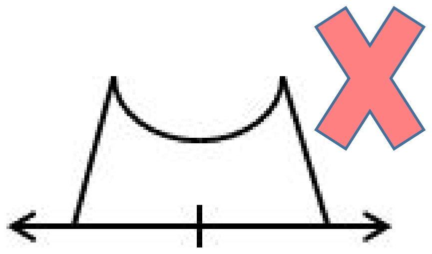
It can be used to calculate power of the signal

$$R_{XX}(\tau) = R_{XX}(-\tau)$$



It is symmetric

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$



At no point it exceeds the average power

Some familiar terms …

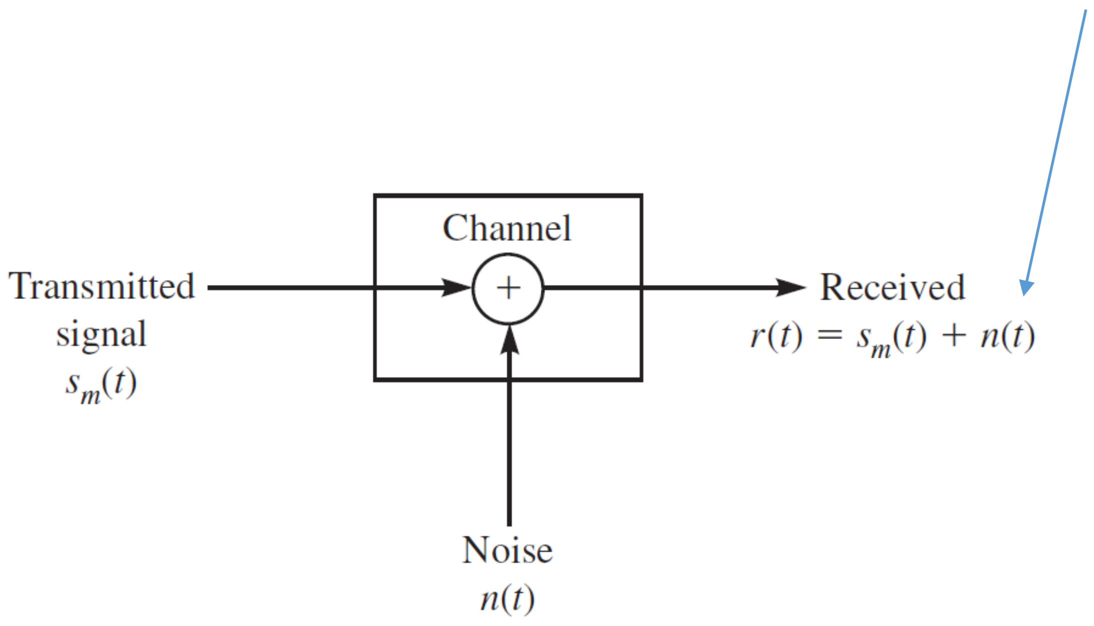
- Probability
- Gaussian Noise
- Fourier Transform
- Power Spectrum
- White Noise

White Noise

- A (stationary) process so random that there is no correlation between any two samples!
  - The samples can follow any distribution (uniform, Gaussian, Poisson etc.), the only condition is that they should have no correlation
  - A white noise process that is characterized by the Gaussian distribution is called **White Gaussian Noise**
- White Noise is a **good approximation** of combined effect of many unrelated noise sources
  - Noise in communication channels is mostly modeled as **Additive White Gaussian Noise (AWGN)**

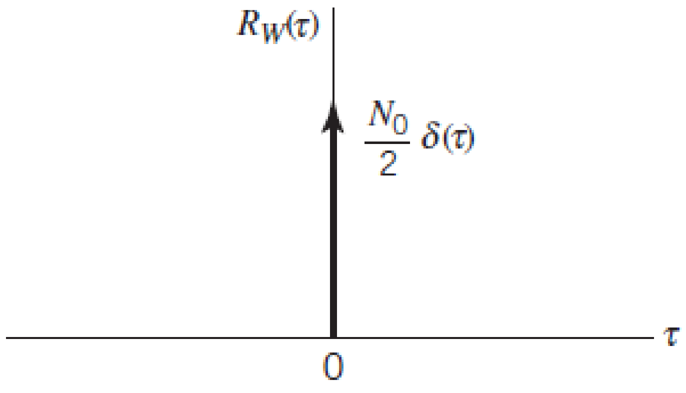
White Noise

- White Noise is a **good approximation** of combined effect of many unrelated noise sources
  - Noise in communication channels is mostly modeled as **Additive White Gaussian Noise (AWGN)**

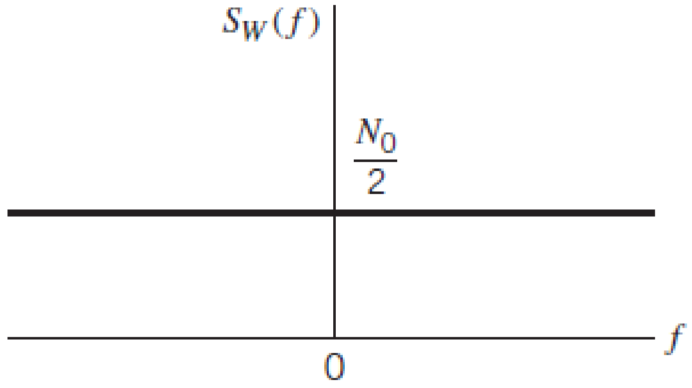


White Noise

- Why “white”?
  - A white noise process has equal contribution from *all* the frequencies (just like white light has equal contribution from all the visible frequencies)



Time description: **No correlation** between any two samples



Frequency description: **equal** contribution from **all frequencies**

Questions?? Thoughts??

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In the last lecture, we saw that …

How do we write the Fourier Transform mathematically?

$$G(f) = \int_{-\infty}^{\infty} g(t)\exp(-j2\pi f t)\,dt$$

$$g(t) = \int_{-\infty}^{\infty} G(f)\exp(j2\pi f t)\,df$$

Shows g(t) written as a "sum" (integral) of complex sinusoids

$$G(f) = \mathbf{F}[g(t)]$$

$$g(t) = \mathbf{F}^{-1}[G(f)]$$

$$\begin{aligned} e^{j\omega t} &= \cos(\omega t) + j\sin(\omega t) \\ \cos(\omega t) &= \tfrac{1}{2}\left(e^{j\omega t} + e^{-j\omega t}\right) \\ \sin(\omega t) &= \tfrac{1}{2j}\left(e^{j\omega t} - e^{-j\omega t}\right) \end{aligned}$$

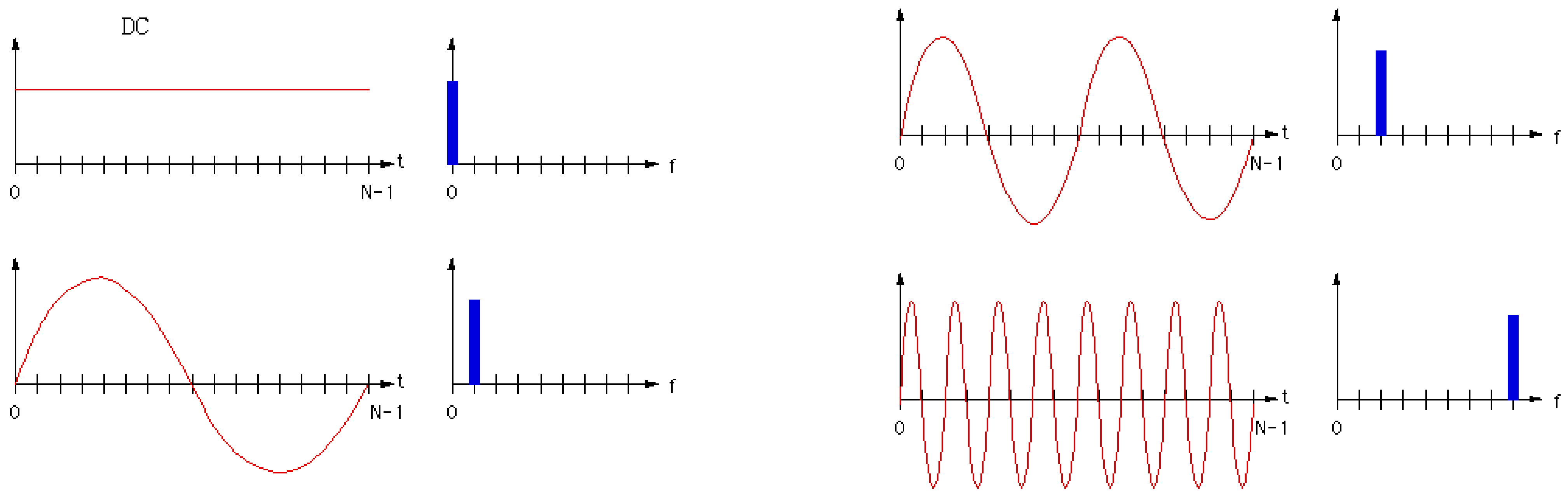
How do we write the Fourier Transform mathematically?

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}$$

$$x[n] = \frac{1}{2\pi}\int_{2\pi} X(e^{j\omega})e^{j\omega n}d\omega$$

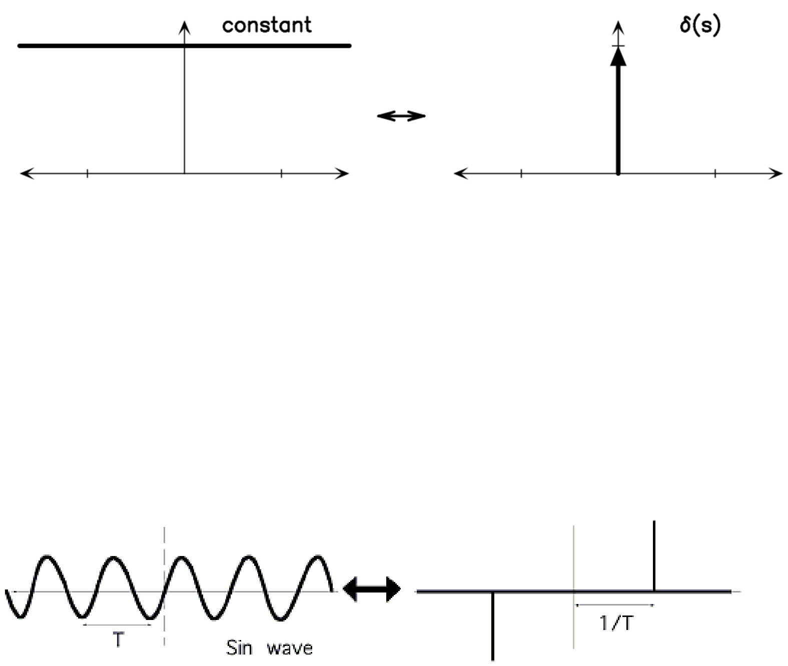
Fourier Transform of discrete time process x[n]

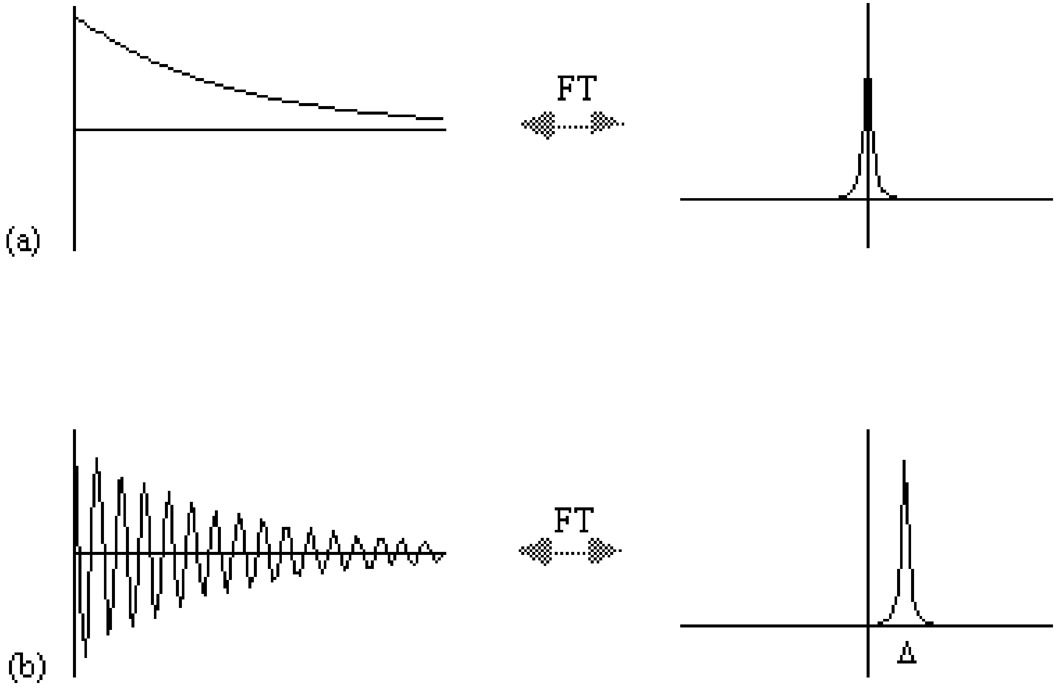
Some Fourier Transforms (Visual)



As frequencies increase, the FT peaks move outwards

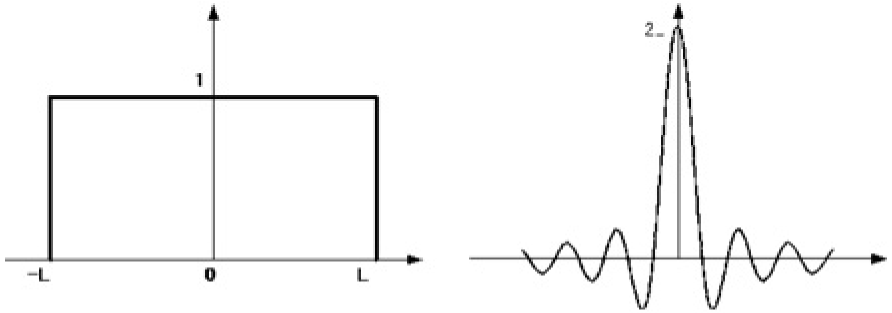
Some Fourier Transforms (Visual)

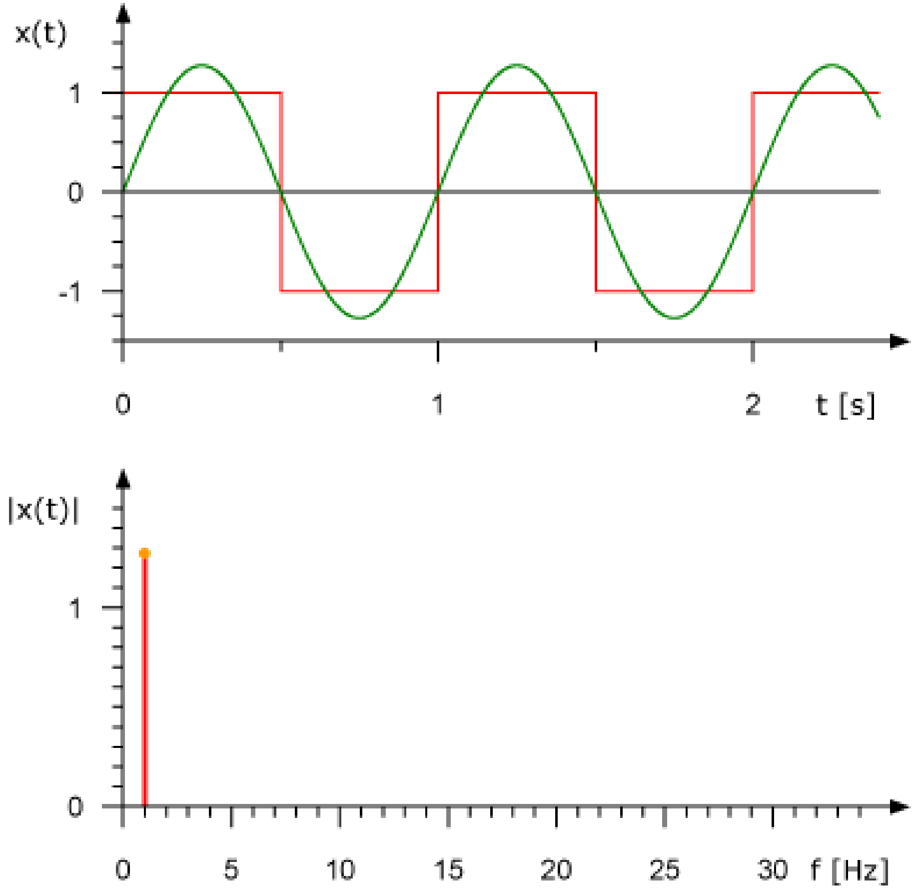




"Damping" causes "spread"

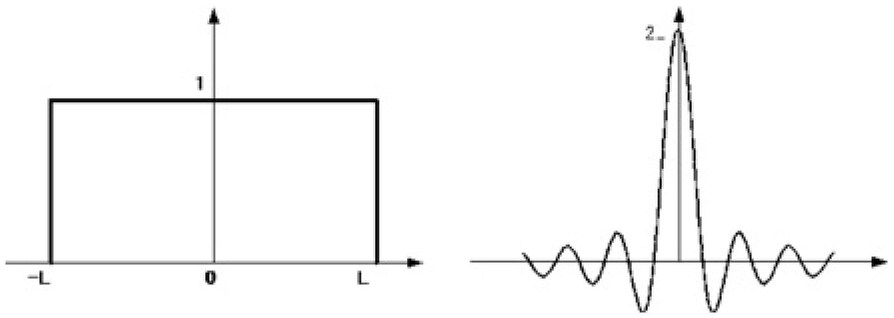
Some Fourier Transforms (Visual)



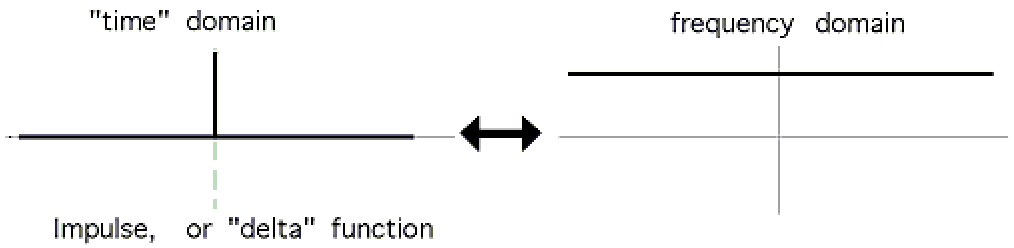


Sharp changes (edges) require a lot of frequencies

Some Fourier Transforms (Visual)



Sharp changes (edges)
require a lot of frequencies



… and an extremely sharp change
(delta) requires ALL the frequencies!!

For a random process it is better to take the Fourier Transform of the autocorrelation function!!

- **Power Spectrum** = Fourier Transform of the autocorrelation function
- Mathematically speaking …

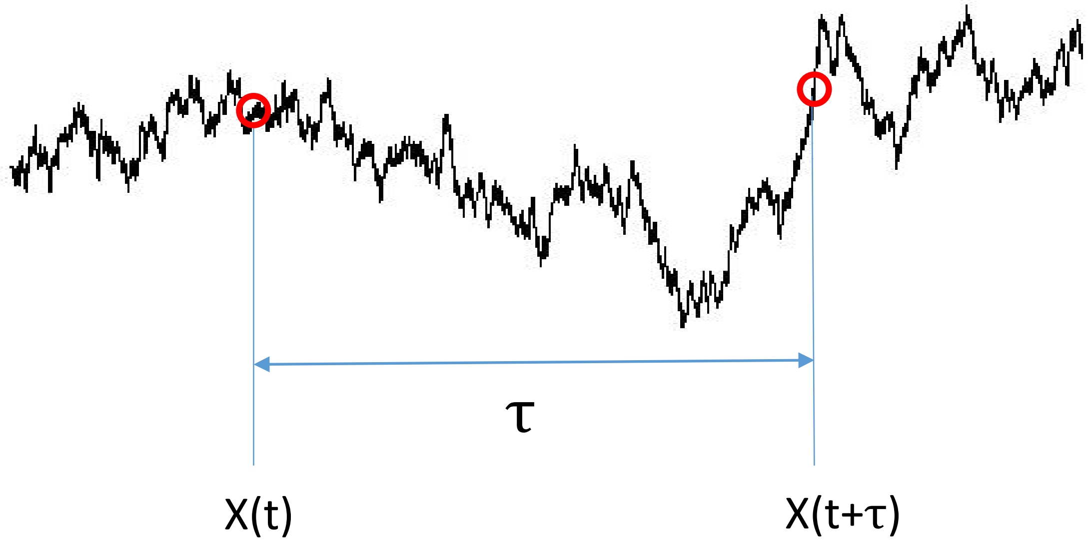
$$S_{XX}(f) = \int_{-\infty}^{\infty} R_{XX}(\tau)\ \exp(-j2\pi f\tau)\ d\tau$$

$$R_{XX}(\tau) = \int_{-\infty}^{\infty} S_{XX}(f)\ \exp(j2\pi f\tau)\ df$$

Calculating the autocorrelation function

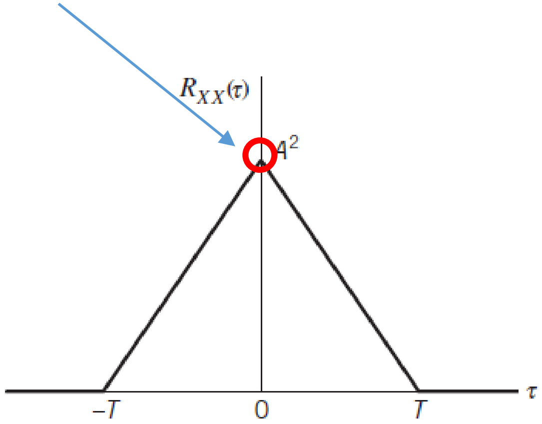
- For a weakly stationary process X(t)

$$R_{XX}(\tau) = \mathbb{E}[X(t+\tau)X(t)]$$



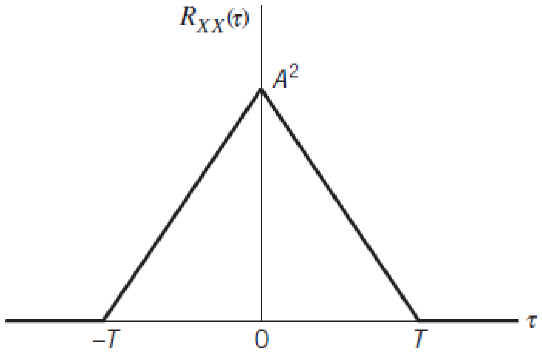
Three important properties of the autocorrelation function

$$R_{XX}(0) = \mathbb{E}[X^2(t)]$$



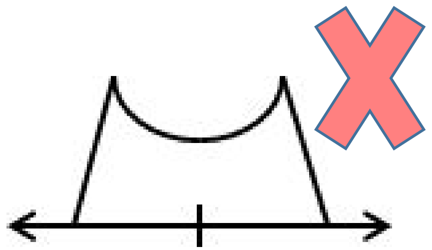
It can be used to calculate power of the signal

$$R_{XX}(\tau) = R_{XX}(-\tau)$$



It is symmetric

$$|R_{XX}(\tau)| \leq R_{XX}(0)$$



At no point it exceeds the average power

Today we shall see …

- What are some of the **properties of the power spectrum**?
- What are some of the **common power spectra** of signals?
- What happens to power spectrum of a signal as it passes **through a channel**?

Five Important properties of Power Spectra of Stationary Processes

Power Spectrum = ***spread of signal power among frequencies***

$$\mathbb{E}[X^2(t)] = \int_{-\infty}^{\infty} S_{XX}(f)\, df$$

(Recall $R_{XX}(0) = \mathbb{E}[X^2(t)]$)

Total area under the power spectrum represents the power of the signal

$$S_{XX}(-f) = S_{XX}(f)$$

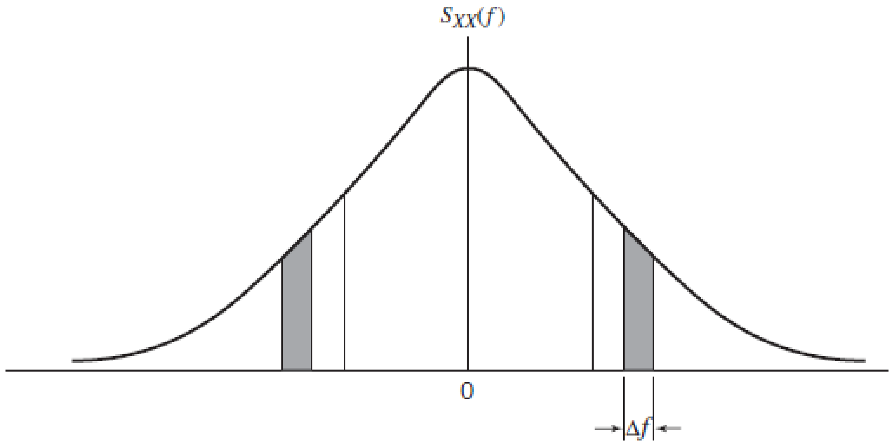
Power Spectra of real processes are always symmetric (can be non-symmetric for complex processes)

$$S_{XX}(f) \geq 0$$

It is always non-negative

Five Important properties of Power Spectra of Stationary Processes

Power Spectrum = ***spread of signal power among frequencies***

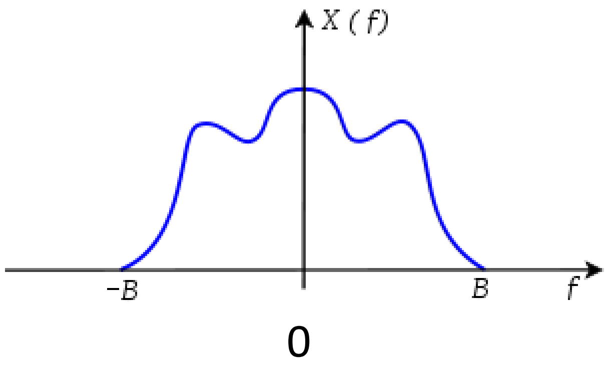


There is no correlation between the frequency components of a Power Spectrum

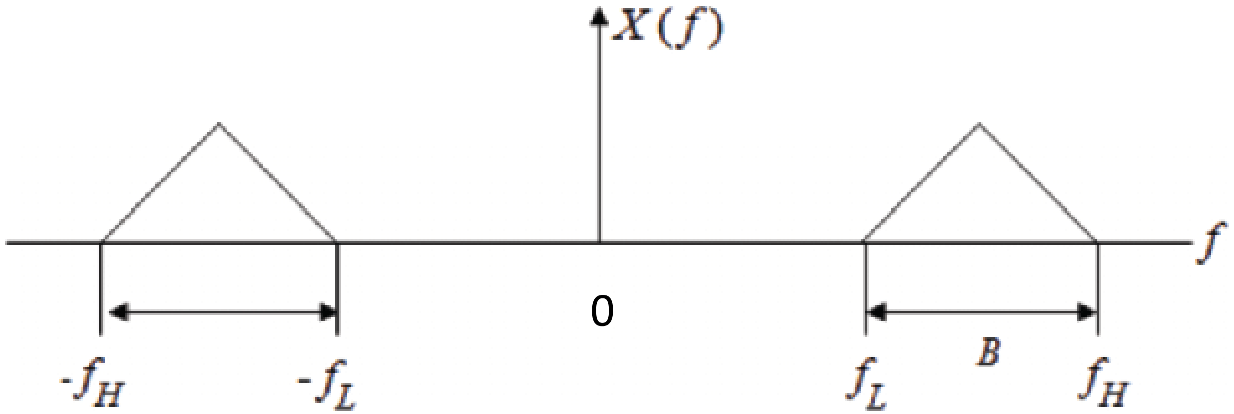
$$S_{XX}(0) = \int_{-\infty}^{\infty} R_{XX}(\tau)\, d\tau$$

The DC component of a Power Spectrum represents the total area under the autocorrelation function

Some common power spectra of signals

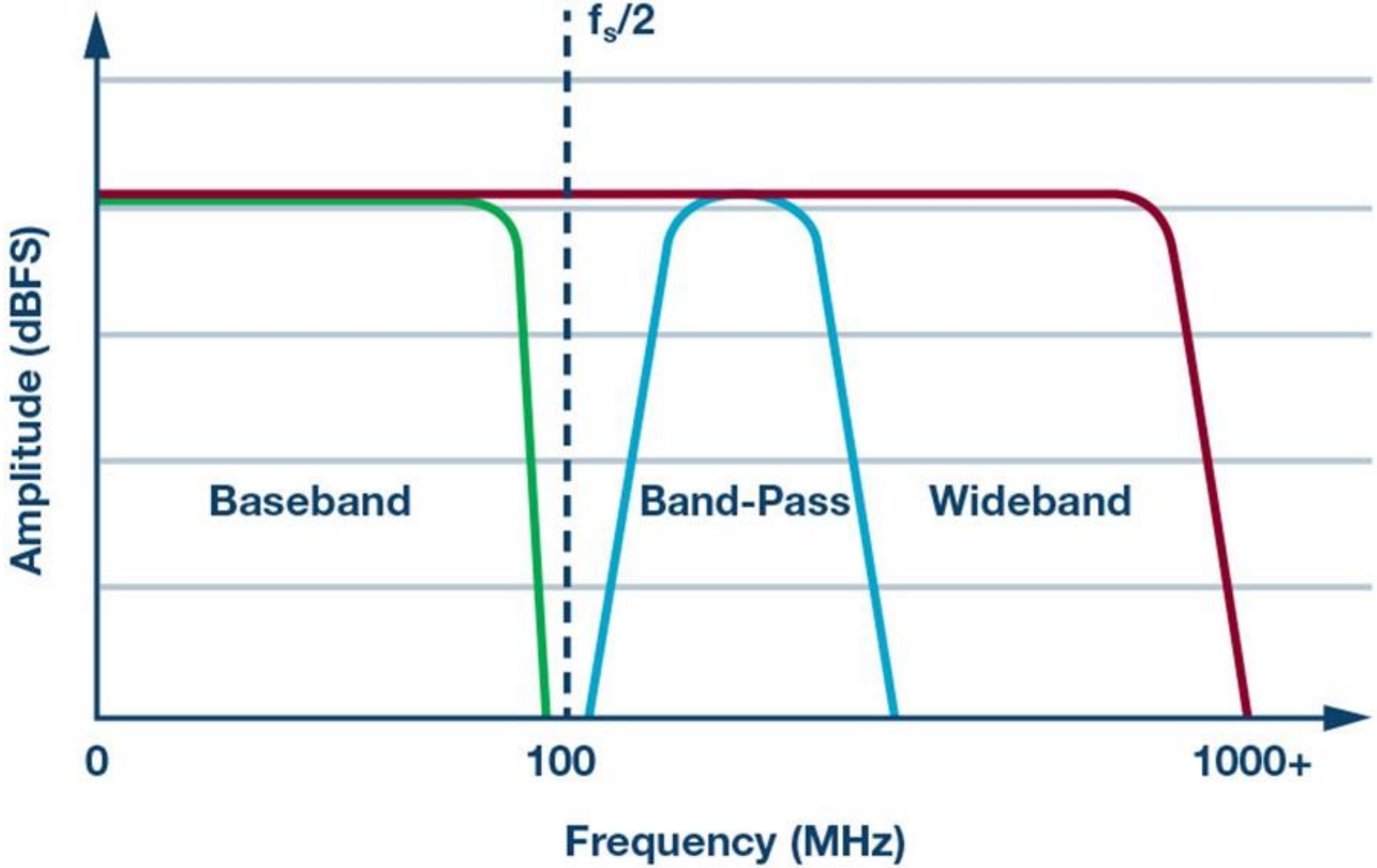


Lowpass or **Baseband** signals have all their power in very low frequencies (around zero)



Bandpass signals have power in bands of high frequencies (away from zero)

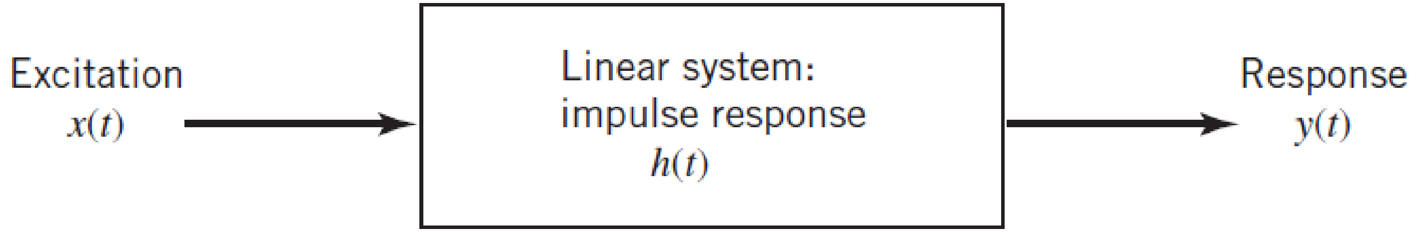
Some common power spectra of signals



What happens to a signal as it passes through a system?

System = ***channel, filter, etc.***

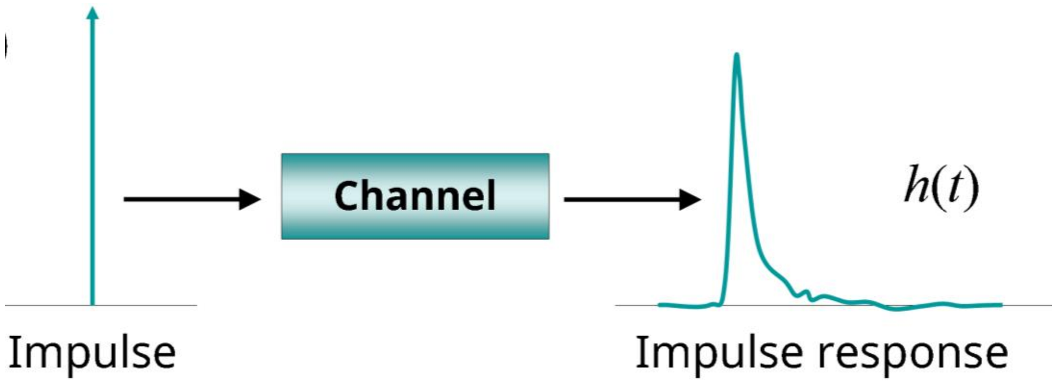
- We shall only consider a special type of systems called **LTI** (linear time-invariant systems).
- We shall only consider weakly stationary signals.



What happens to a signal as it passes through a system?

System = ***channel, filter, etc.***

- An LTI system is fully characterized by its **Impulse Response** (IR)
  - IR = what output the system gives when the input is an impulse (a theoretical sharp pulse)

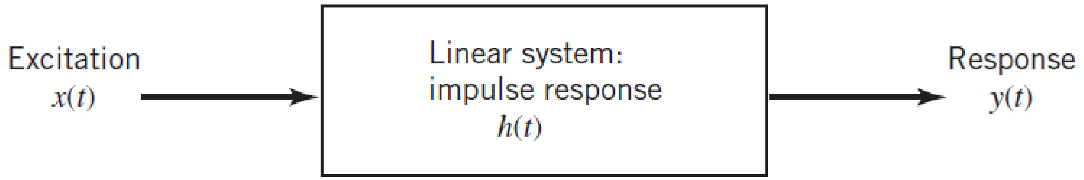


What happens to a signal as it passes through a system?

System = ***channel, filter, etc.***

- If we know the **impulse response** of an LTI system we can find its output to any signal by using **convolution**

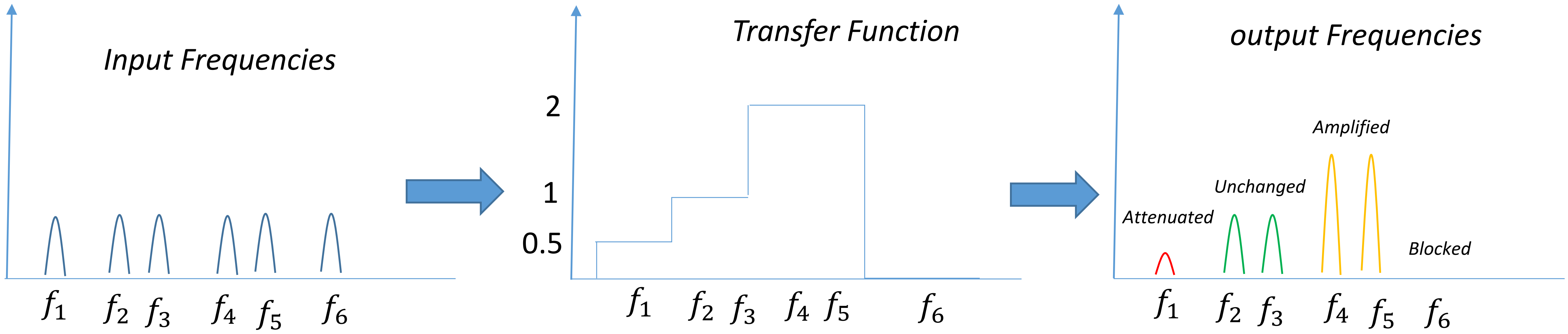
$$y(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)\,d\tau$$



What happens to a signal as it passes through a system?

System = ***channel, filter, etc.***

- LTI system can also be fully characterized by its **Transfer Function** (TF)
  - TF = Fourier Transform of Impulse Response
  - TF = what the system does to different frequencies of the signal (e.g., blocks, allows unchanged, amplifies, attenuates)

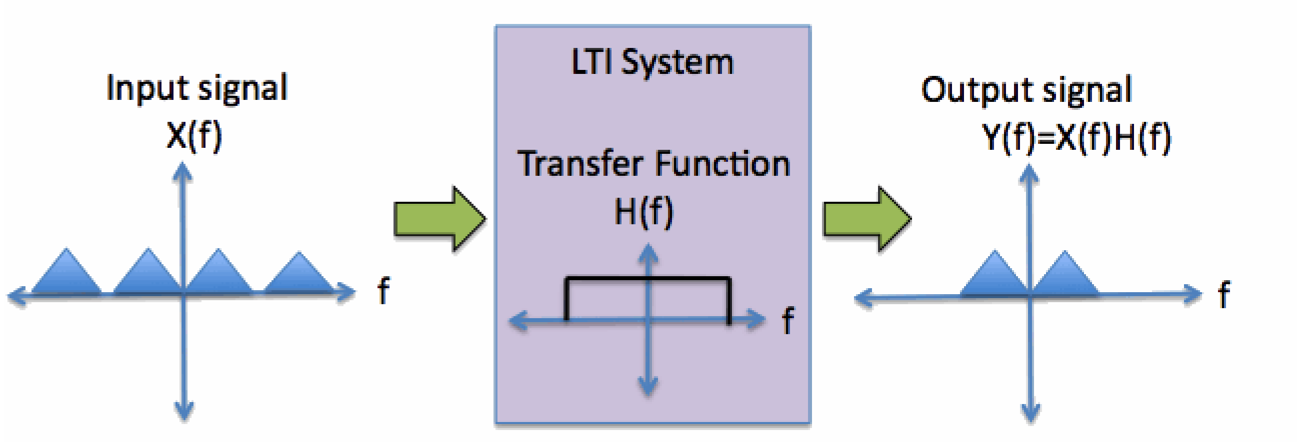


What happens to a signal as it passes through a system?

System = ***channel, filter, etc.***

- If we know the **Transfer Function** of an LTI system we can find what it does to the frequencies of a **deterministic signal** by using simple multiplication in frequency domain

$$Y(f) = H(f)X(f)$$



What happens to a signal as it passes through a system?

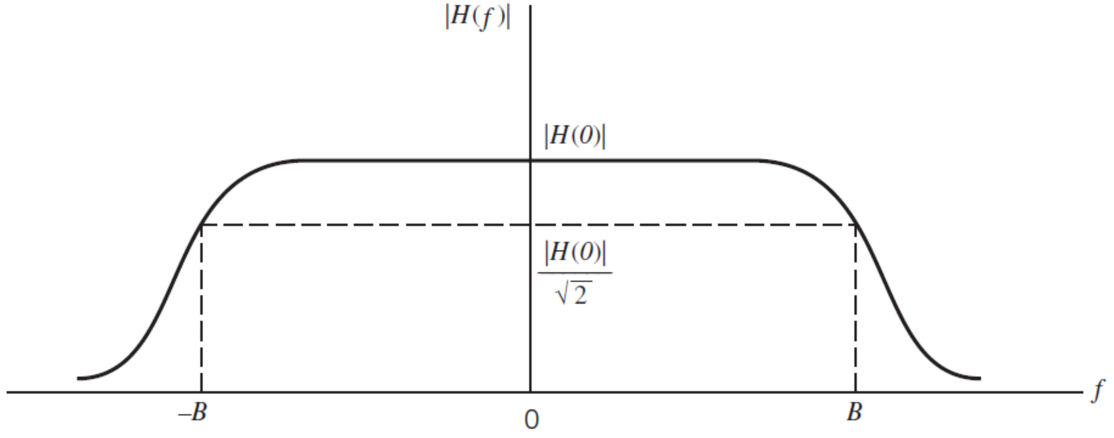
System = ***channel, filter, etc.***

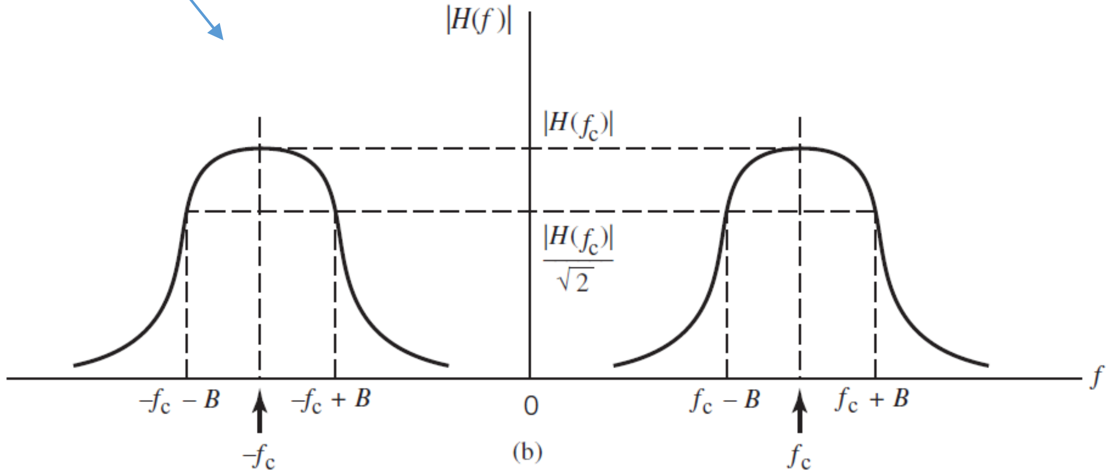
- If we know the **Transfer Function** of an LTI system we can find what it does to the **power spectrum** of weakly stationary **random signal** by

$$S_{YY}(f) = |H(f)|^2 S_{XX}(f)$$

$$|H(f)|^2 = H(f)H^*(f)$$

Low-Pass and Band-Pass Channels





Questions?? Thoughts??

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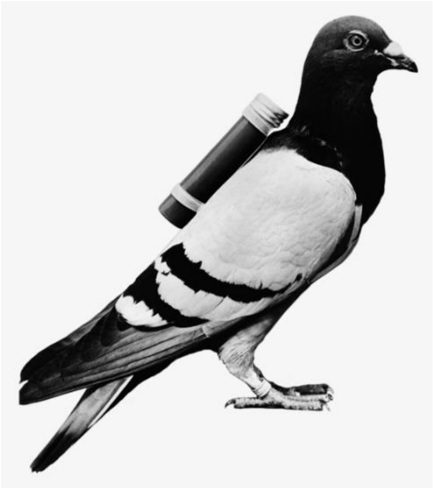
Modulation : *message on a carrier*

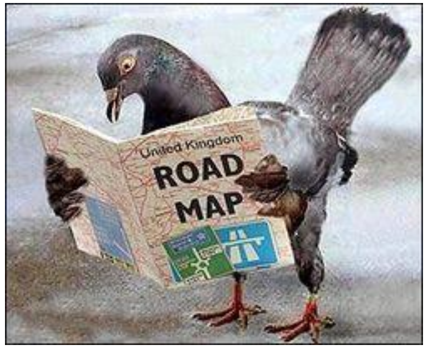
- **Modulation** is the process of placing the message on a carrier so that it can be carried (transmitted) to the intended receiver
  - **Message** = the actual information we want to send
  - **Carrier** = some signal/symbol which can represent and carry the information

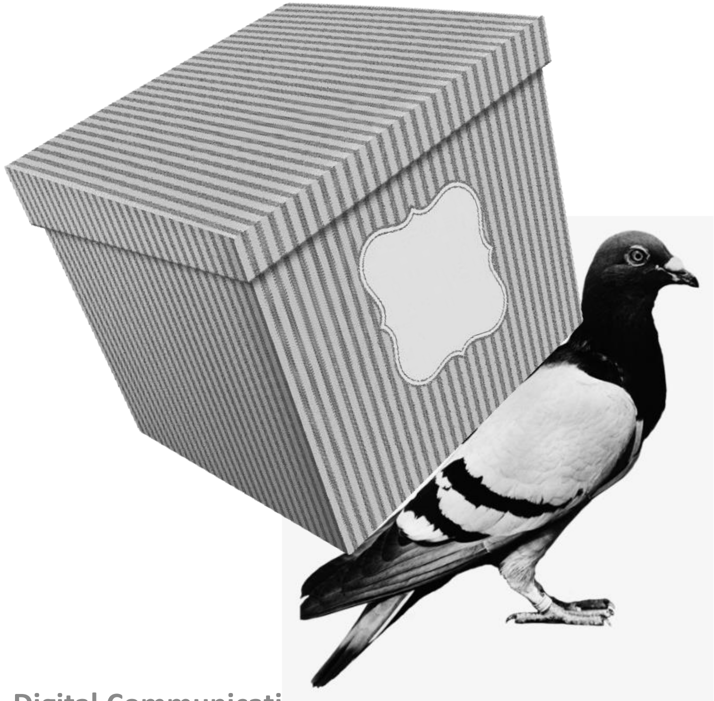


Demodulation

- **Demodulation** is the opposite of modulation, i.e., extracting the message from the carrier

What are some of the properties a carrier should have?



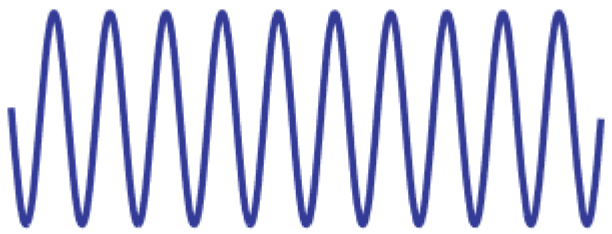


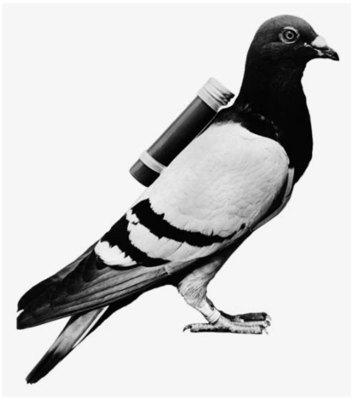
What are some of the properties a carrier should have?

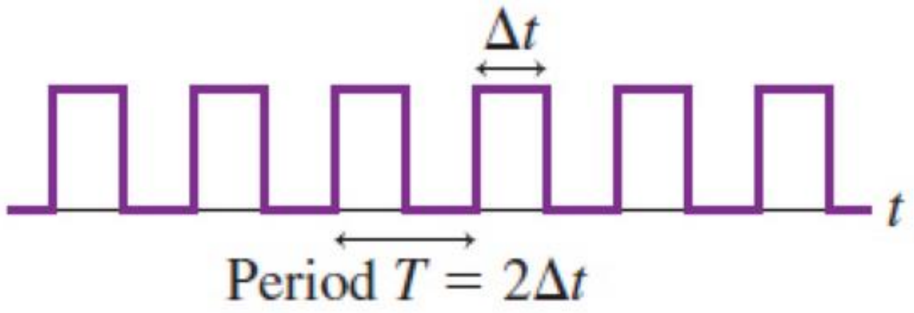
- It should be **able to load (or represent) the information** somehow
- It should be **able to carry the message along the desired path** and distance (and should be better at it than the message itself)
- It should be **able to handle the amount of information** in the message
- There should be a **way for the receiver to take (extract) the information** from it somehow
- And **a "good" carrier** should provide high-quality and efficient service (low-cost, low-power, fast, reliable, safe etc.)

Common Carriers: *Waves and Pulses*

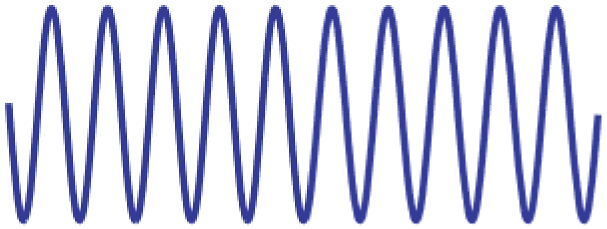
- A sinusoidal wave can carry information
  - How? Coming up!
- A train of pulses can carry information
  - How? Coming up!



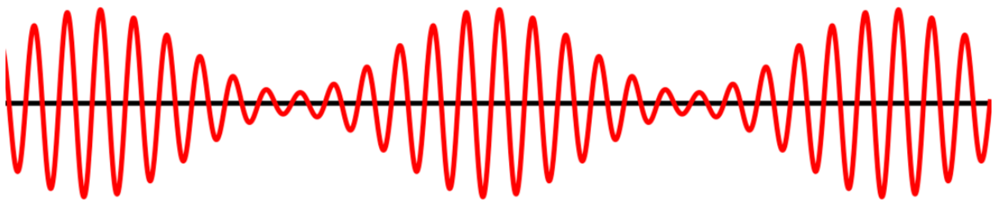


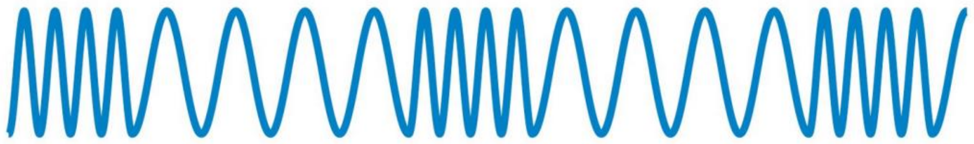


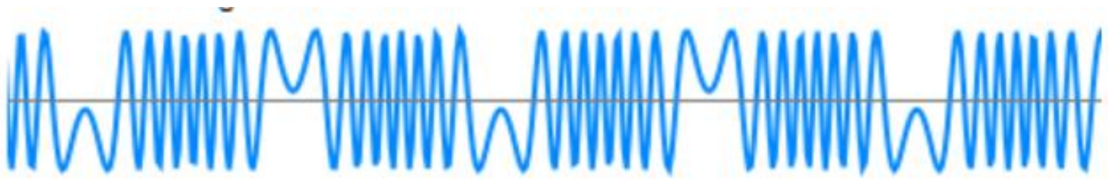
Waves: Amplitude, Frequency, and Phase



- Consider a sinusoidal wave
  - $A cos(2\pi f t + \phi)$
- What can you “tune” here to change the wave?





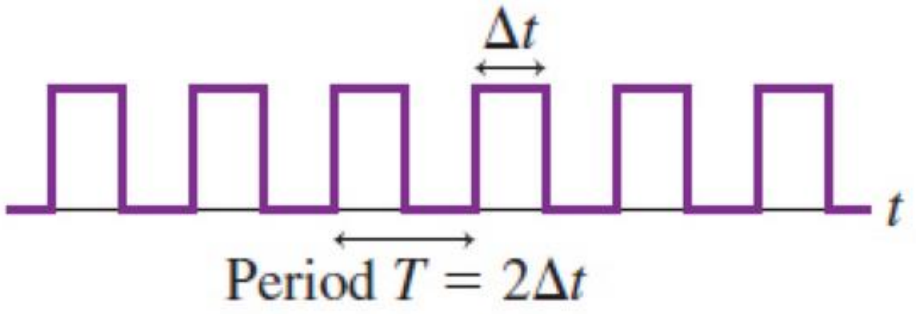


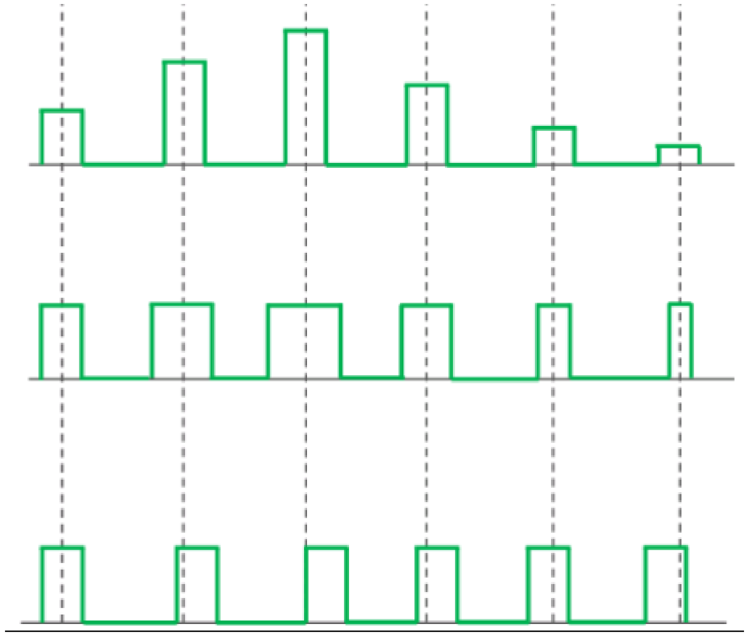
Waves: Amplitude, Frequency, and Phase

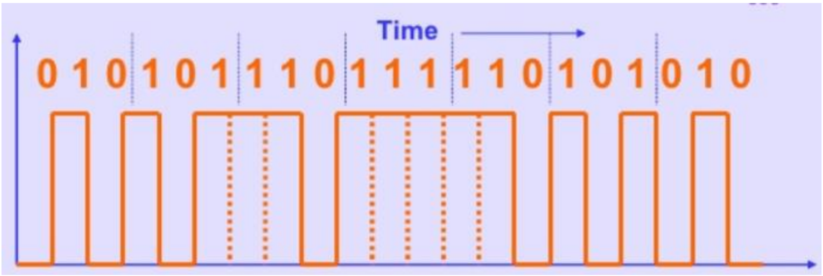
- Consider a sinusoidal wave
  - $A\cos(2\pi f t + \phi)$
- What can you "tune" here to change the wave?
  - You could play with its **amplitude** (A)
  - You could play with its **frequency** (f)
  - You could play with its **phase** (ϕ)
  - Or, you could play with any two or all of the above at the same time
- **Note: all of the above "changes" can represent information!!**
  - e.g., high frequency = 1, low frequency = 0 etc.

Pulses: Amplitude, Position, Width, and Pattern

- Consider a train of pulses
- What can you “tune” here?







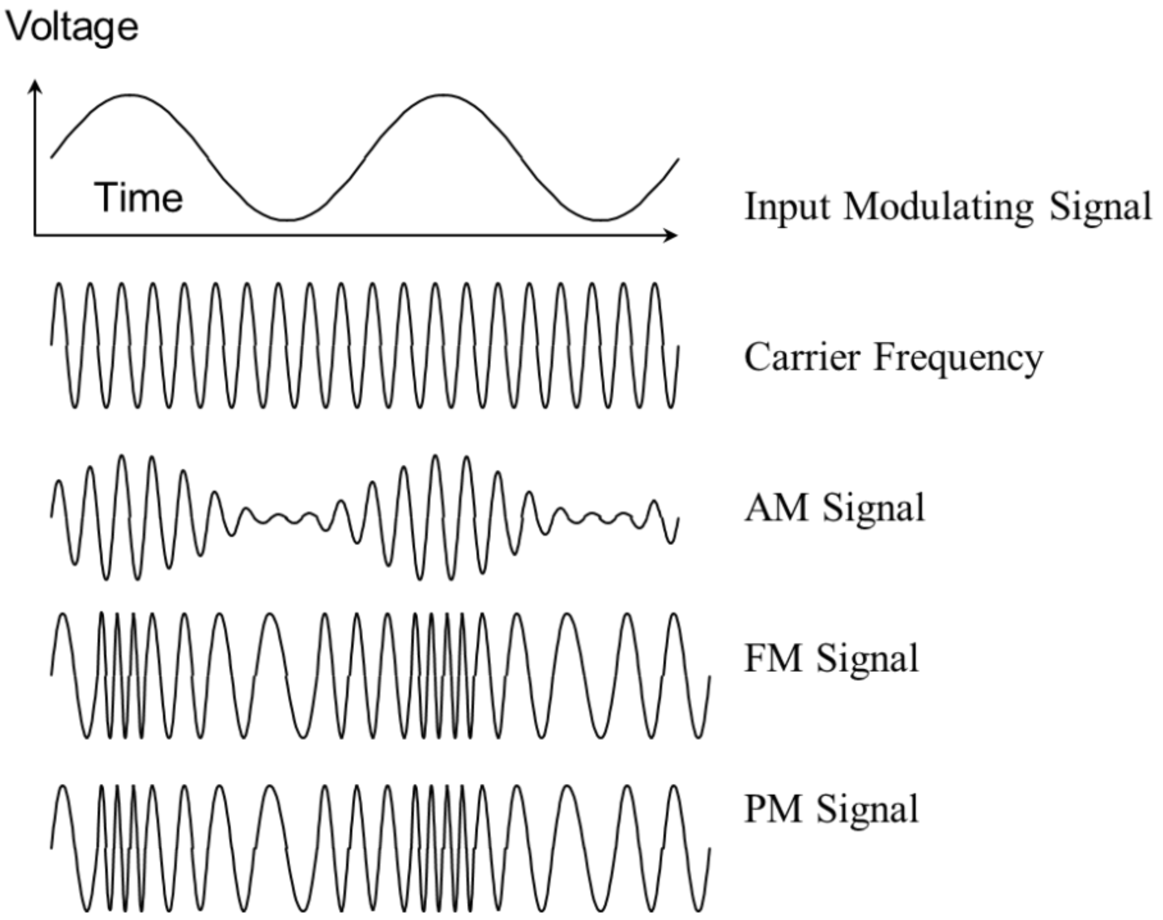
Pulses: Amplitude, Position, Width, and Pattern

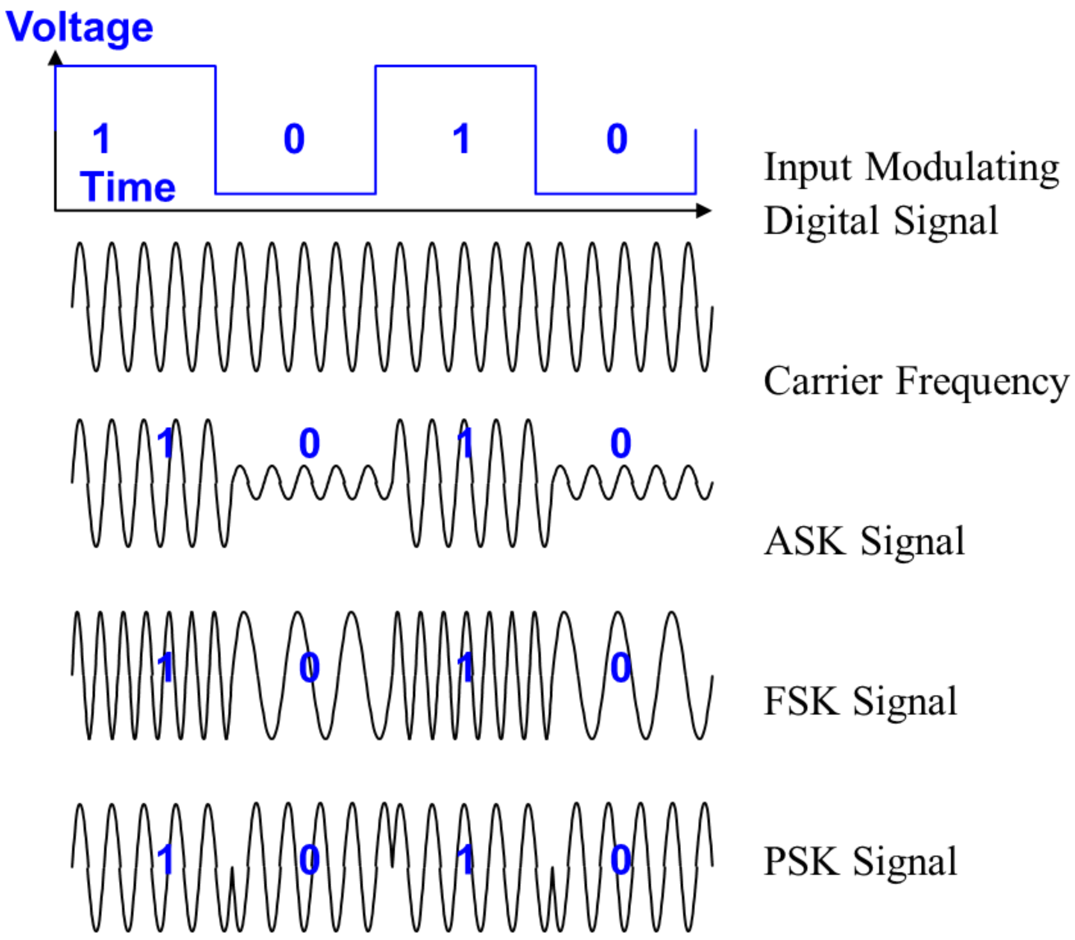
- Consider a train of pulses
- What can you "tune" here?
  - You could play with the **amplitudes** of the pulses
  - You could play with the **positions** of the pulses
  - You could play with the **widths** of the pulses
  - Or, you could "encode" information in some special **patterns** of the pulses
- **Note: all of the above "changes" can represent information!!**
  - e.g., pulse = 1, no pulse = 0

Message Types: Analog and Digital

- We saw that both waves and pulses can be used to carry information (messages)
- Next, we see the **two common forms a message takes**
- Message in **analog** form
- Message in **digital** form

Message Types: Analog and Digital





Waves

Message Analog

- Amplitude Modulation (AM)
- Frequency Modulation (FM)
- Phase Modulation (PM)
- Analog QAM

Message Digital (or Digitized)

- Amplitude Shift Keying (ASK)
- Frequency Shift Keying (FSK)
- Phase Shift Keying (PSK)
- Digital QAM

Pulses

Message Analog

- Pulse Amplitude Modulation
- Pulse Width Modulation
- Pulse Position Modulation

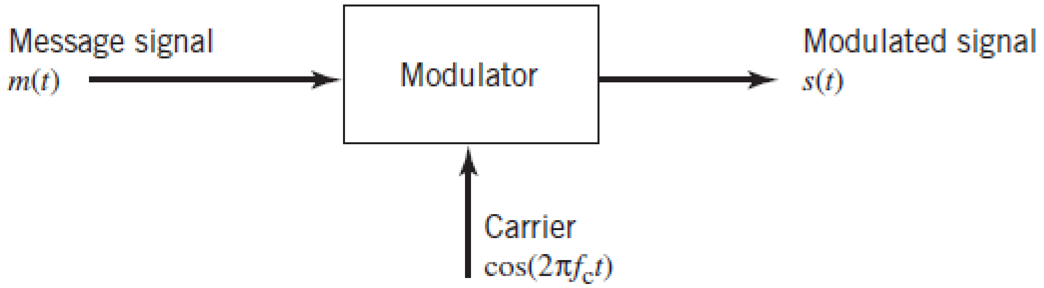
Message Digital (or Digitized)

- Pulse Code Modulation

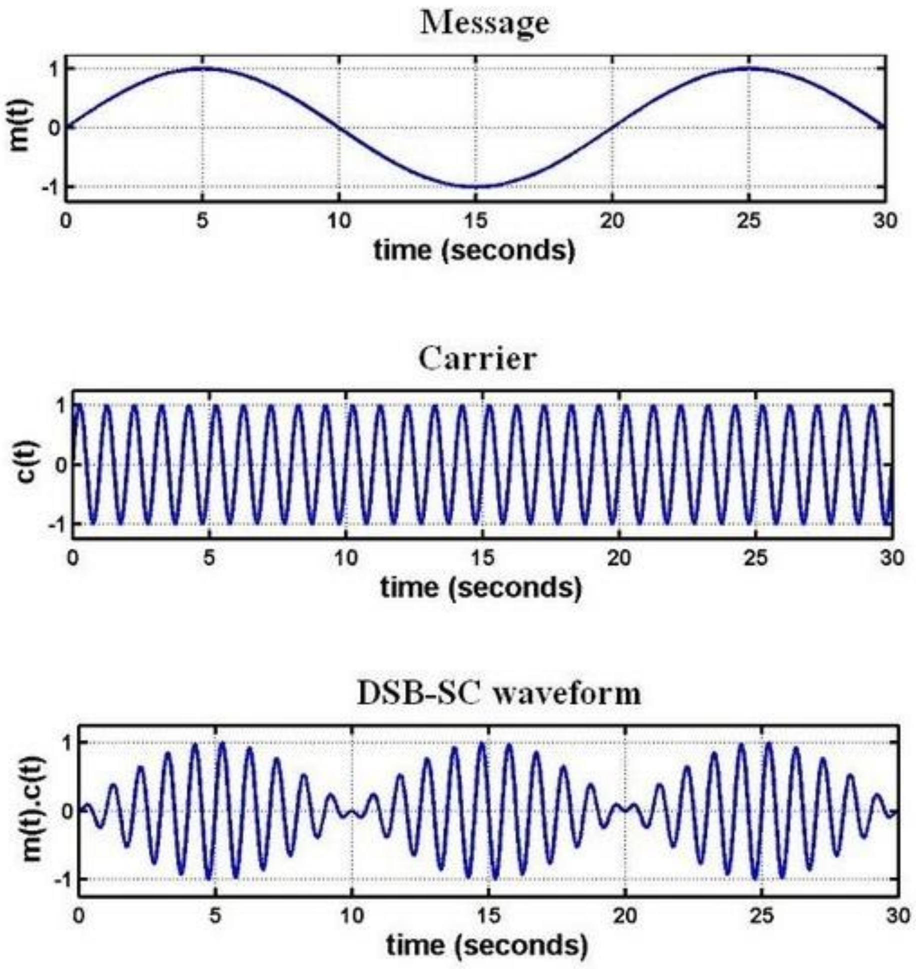
Mathematically: the message and the carrier

- We look at a common form of amplitude modulation to identify the parts of a modulation/demodulation scheme
- We also look at power spectra of message and modulated signals
- How to demodulate? Two approaches: coherent (multiply by local wave and low-pass), non-coherent (simple low-pass: detect envelope)
  - Why use coherent then?

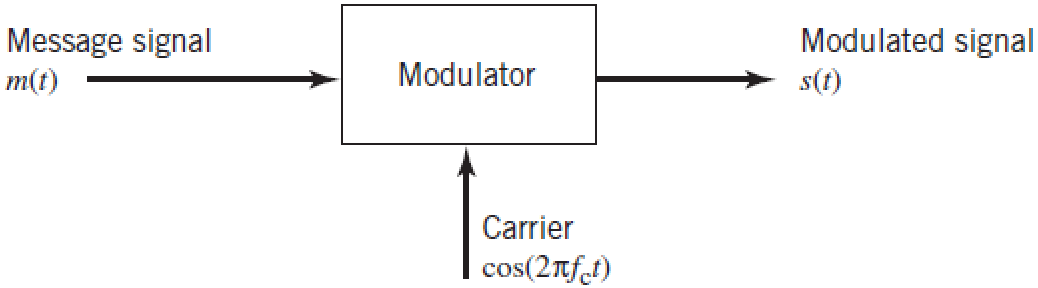
Mathematically: the message and the carrier



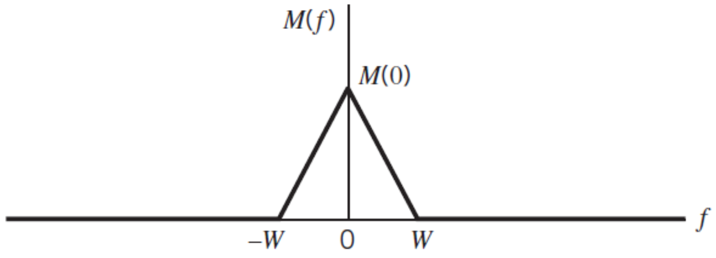
$$s(t) = m(t)\cos(2\pi f_c t)$$

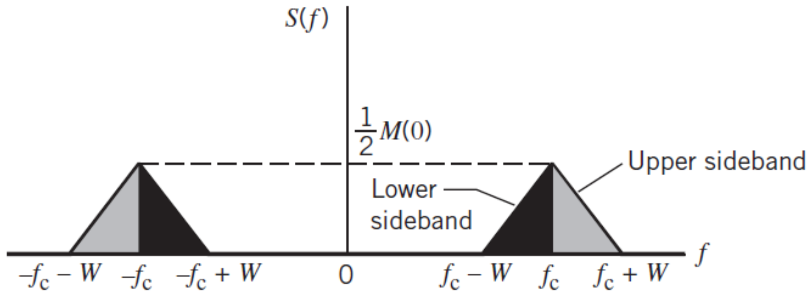


Mathematically: the message and the carrier



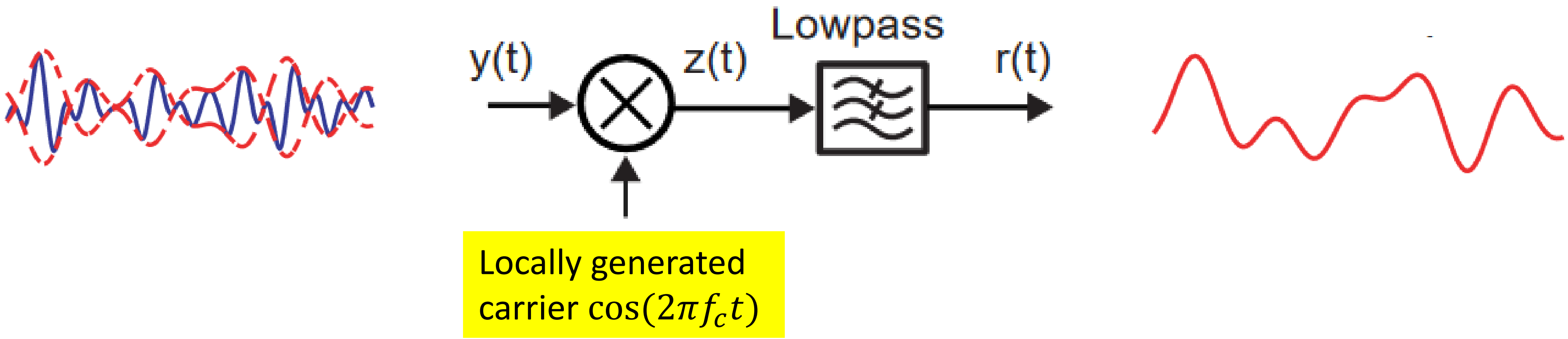
$$s(t) = m(t)\cos(2\pi f_c t)$$





Coherent Demodulation

- Multiply the received signal with a locally generated carrier which is (ideally) synchronized with the carrier used by the modulator.
- Pass the resulting signal through a lowpass filter to extract the message.



Mathematically: *Coherent Demodulation*

Assuming noise-free transmission, received signal is

$$y(t) = s(t) = m(t)\cos(2\pi f_c t)$$

Multiplication with locally generated carrier results in

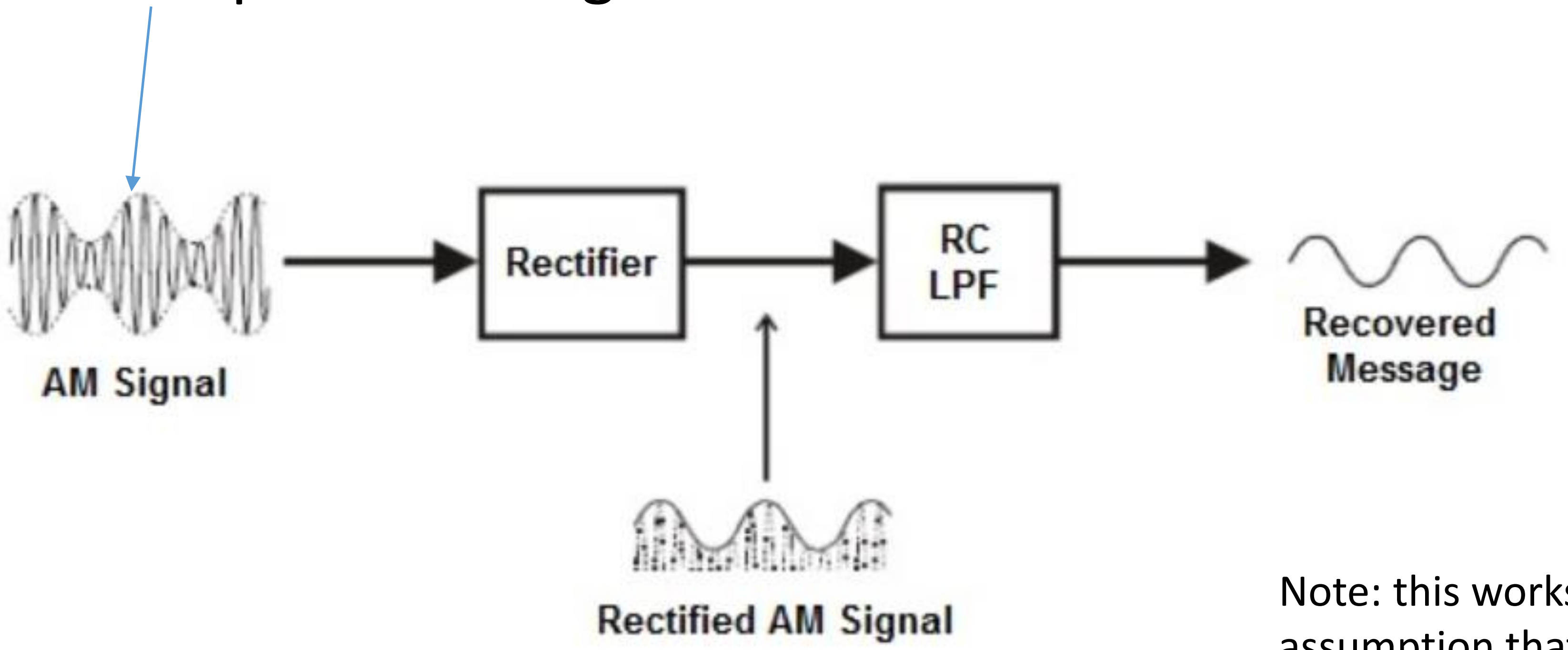
$$y(t)\cos(2\pi f_c t) = m(t)\cos^2(2\pi f_c t) = \frac{1}{2}m(t) + \frac{1}{2}\cos(4\pi f_c t)$$

Passing through a lowpass filter gets rid of the second term, leaving us with the message at the receiver

$$r(t) = m(t)$$

Non-Coherent Demodulation

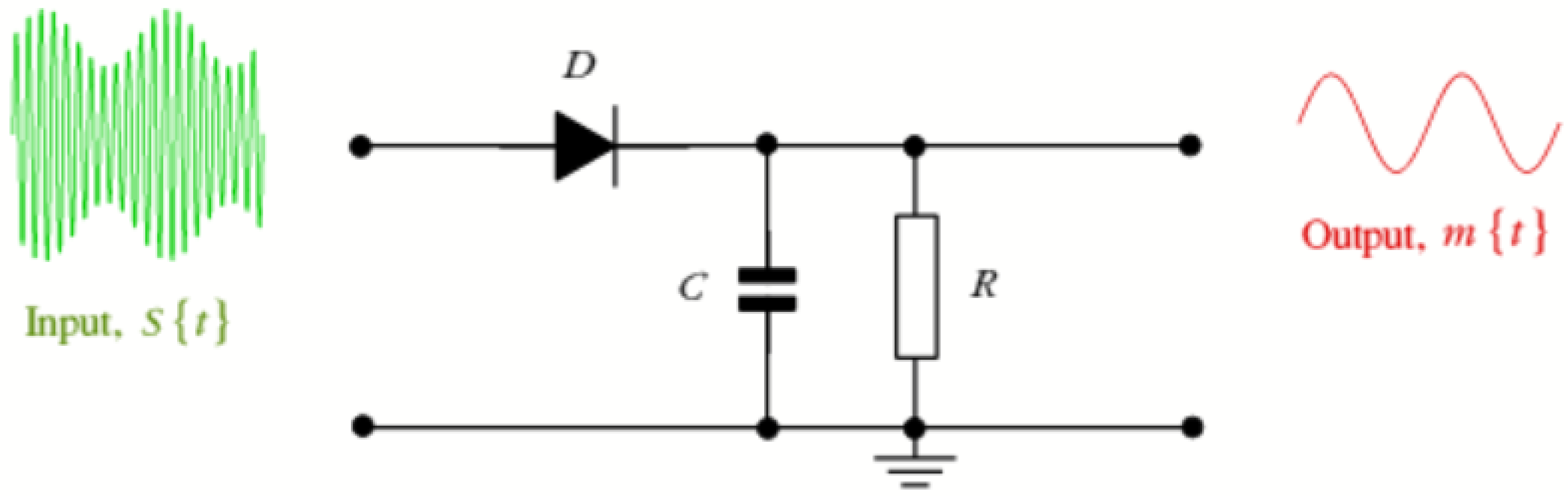
- In non-coherent demodulation we do not generate a local version of the carrier and use an **envelope detector** instead to extract the slowly varying “envelope” of the signal.



Note: this works under the assumption that the signal is not *over-modulated*.

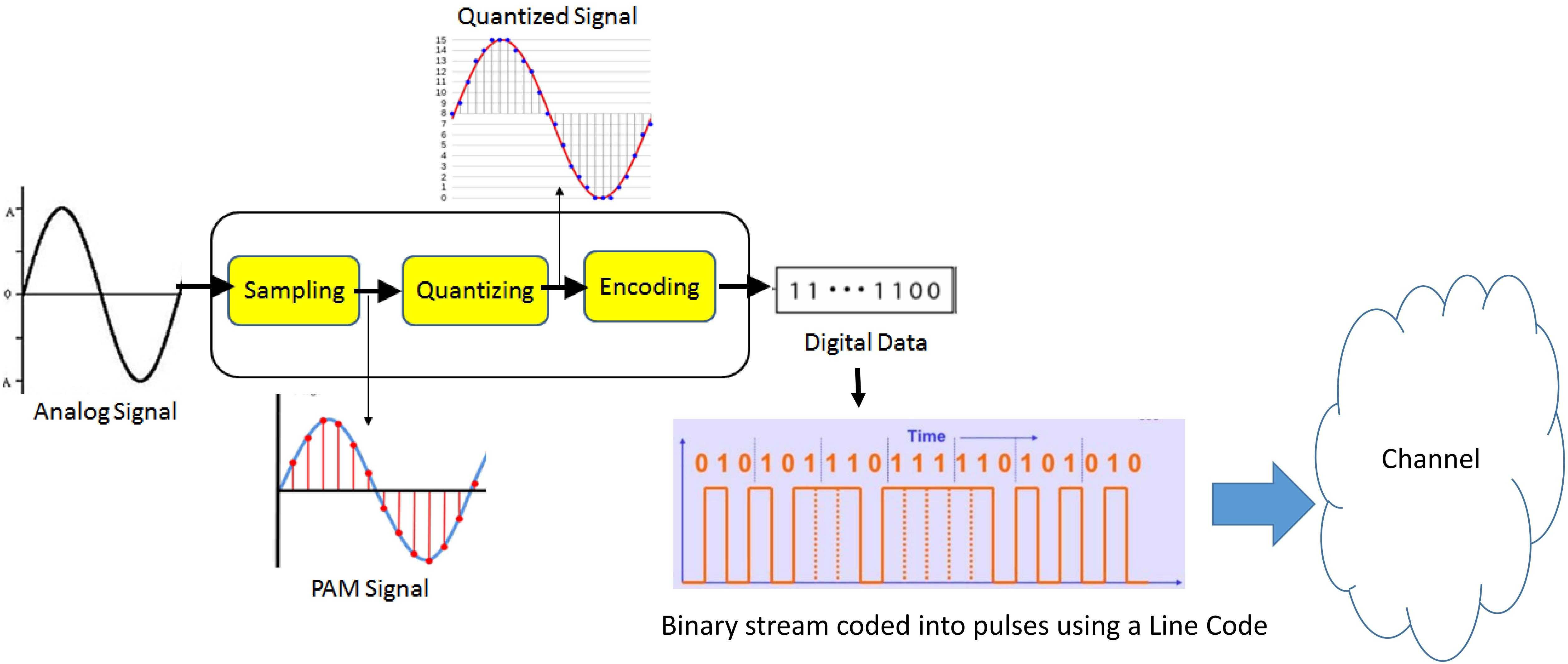
Non-Coherent Demodulation

- In non-coherent demodulation we do not generate a local version of the carrier and use an **envelope detector** instead to extract the slowly varying "envelope" of the signal.



Pulse Code Modulation (PCM)

- In PCM we take an analog signal (message) and transmit it in form of digital pulses.

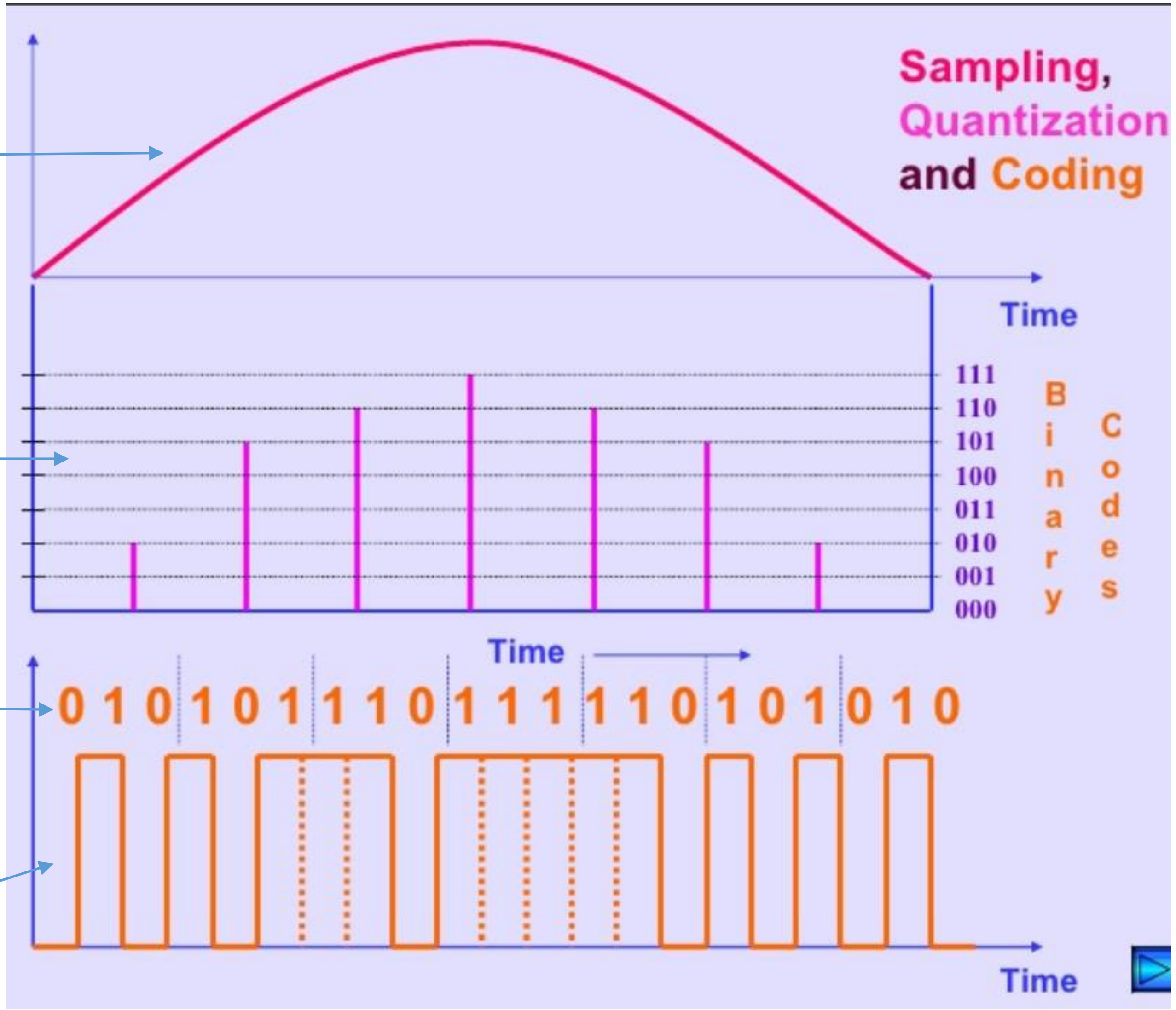

Quantized Signal
Sampling
Quantizing
Encoding
11 • • • 1100
Digital Data
Analog Signal
PAM Signal
Time
0 1 0 1 0 1 1 1 0 1 1 1 1 1 0 1 0 1 0 1 0
Binary stream coded into pulses using a Line Code
Channel

Message in analog form

Digitized message (sampled and quantized)

Message converted to binary stream

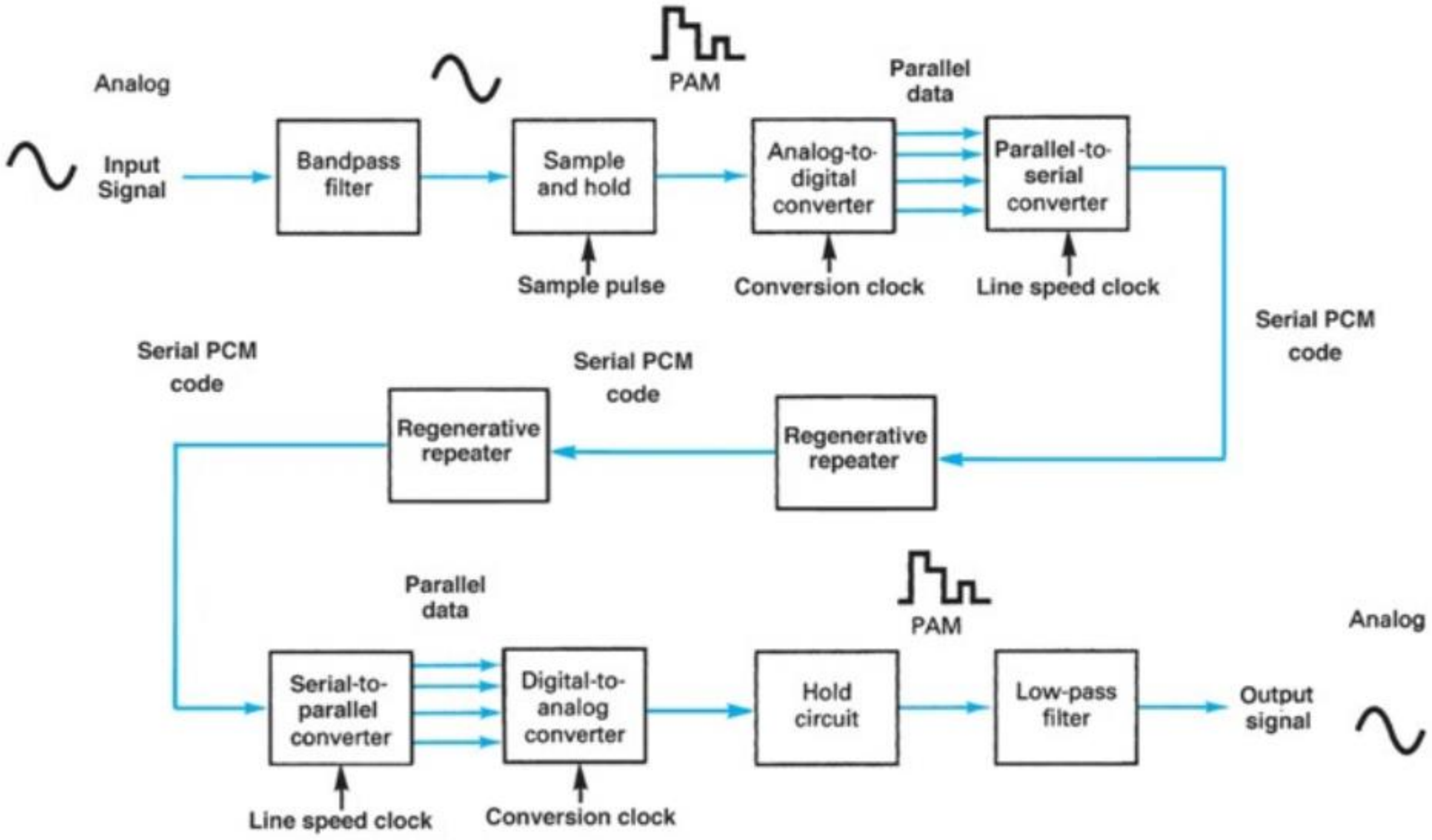
Binary stream converted to pulses



Pulse Code Modulation (PCM)

- In PCM we take an analog signal (message) and transmit it in form of digital pulses.
- Steps involved are
  - **Sampling**: convert analog signal to discrete signal
  - **Quantization**: quantize the collected samples to fixed levels
  - **Encoding**: represent the quantized samples in binary
    - Line Codes: choose how 0s and 1s will be represented in pulses

PCM Transmitter



PCM Receiver

masi

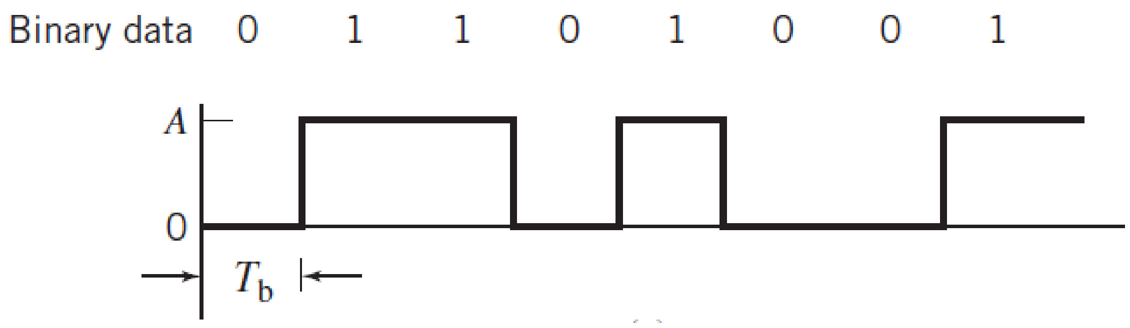
How to Represent 0s and 1s? *Line Codes*

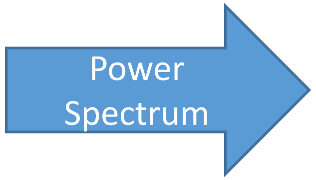
- PCM (and its variants) need to represent 0s and 1s electrically for transmission of binary streams
- The schemes used for representing 0s and 1s are called Line Codes
- Five types of Line Codes are commonly used
  - Unipolar NRZ
  - Polar NRZ
  - Unipolar RZ
  - Bipolar RZ
  - Manchester

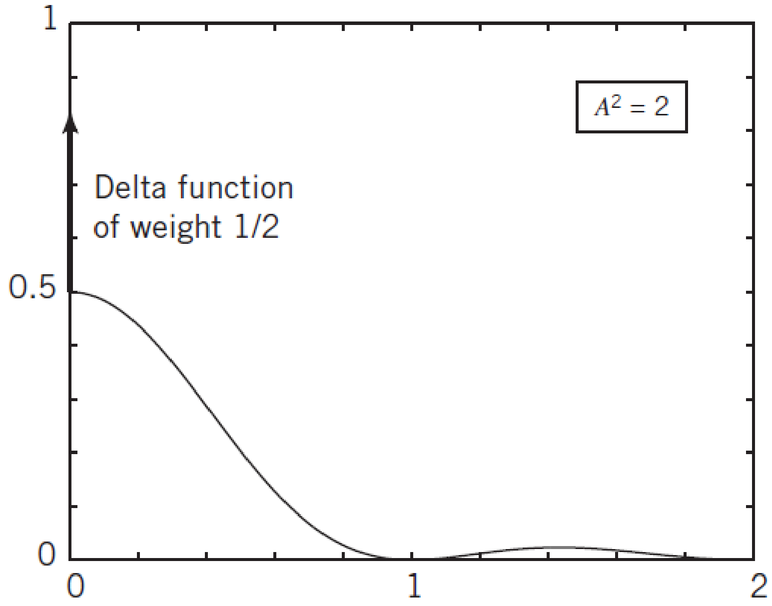
Terminology: NRZ vs RZ, Unipolar vs Polar

- Nonreturn-to-Zero (NRZ)
  - Class of Line Codes where the pulse level does not return to (and stay at) zero level during one symbol (bit)
- Return-to-Zero (RZ)
  - Class of Line Codes where the pulse level returns to (and stays at) zero level for some time during one symbol (bit)
- Unipolar
  - Class of Line Codes which use only one polarity for pulse amplitude (e.g., 0 or +10, but not negative)
- Polar
  - Class of Line Codes which both positive and negative polarities for the pulse amplitude (e.g., +10 and -10)

1. Unipolar NonReturn-to-Zero (NRZ)





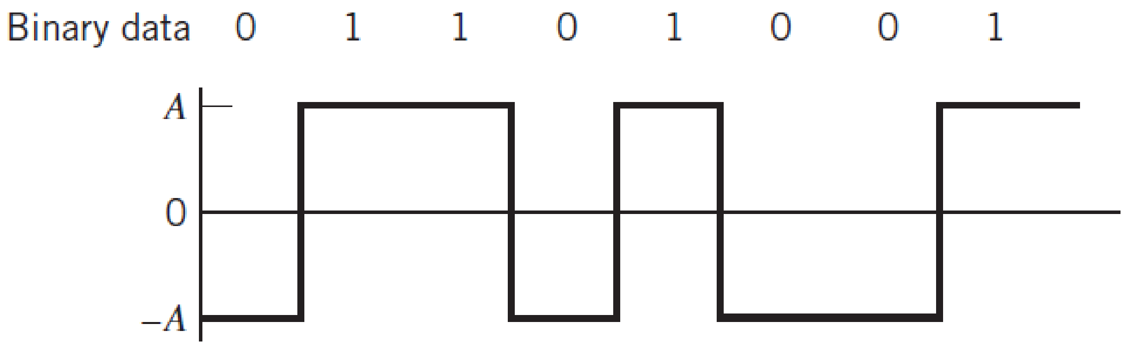


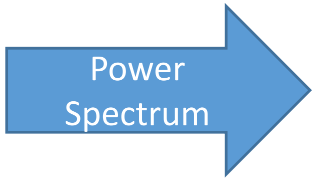
0 = no pulse
1 = pulse of amplitude *A*

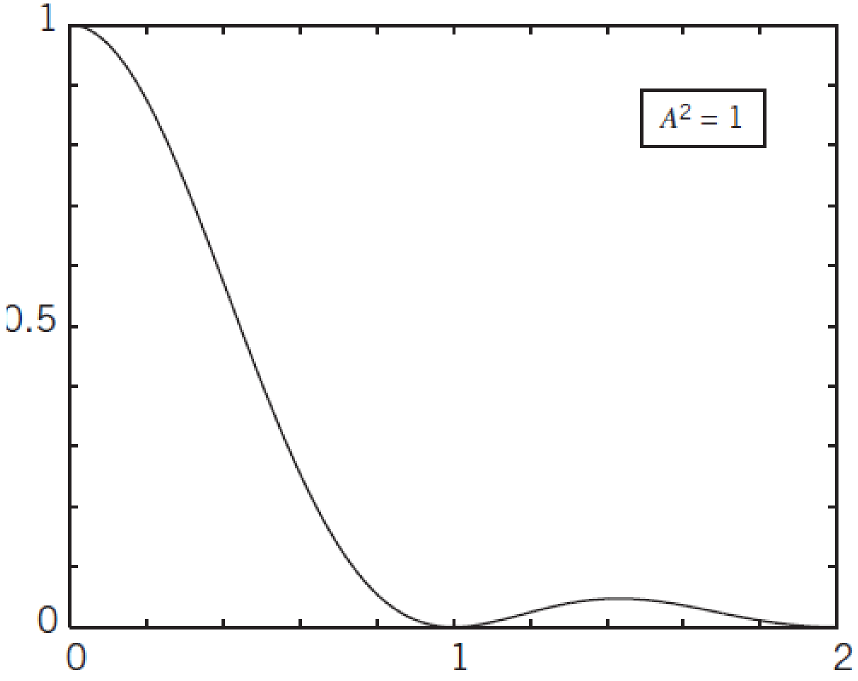
Advantages:
- Simple

Disadvantages:
- Transmits DC level as well (waste of power)

2. Polar NonReturn-to-Zero (NRZ)





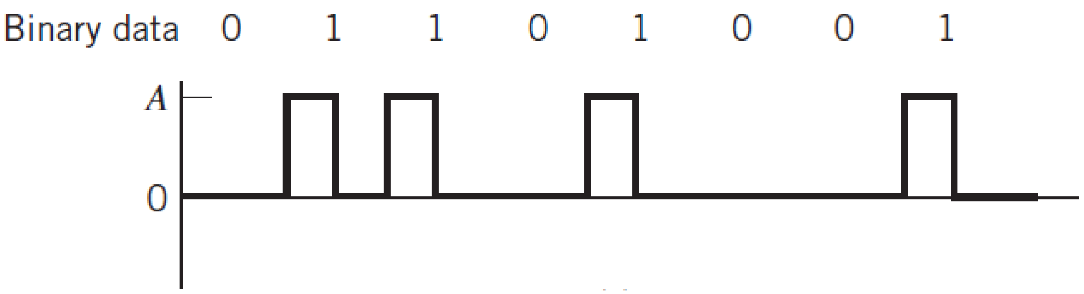


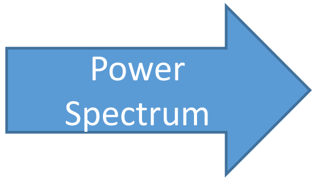
0 = pulse of amplitude -*A*
1 = pulse of amplitude +*A*

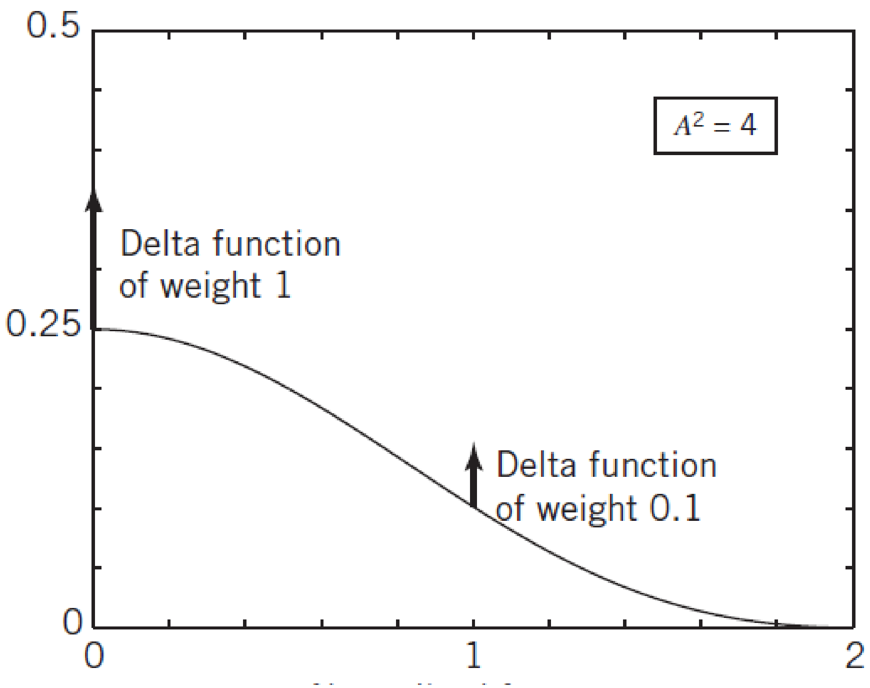
Advantages:
- Simple

Disadvantages:
- Large power near DC (waste of power)

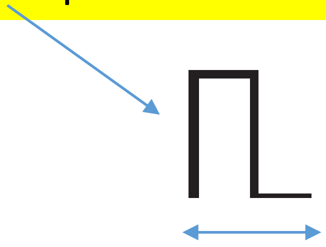
3. Unipolar Return-to-Zero (RZ)







0 = no pulse
1 = pulse that is half symbol wide

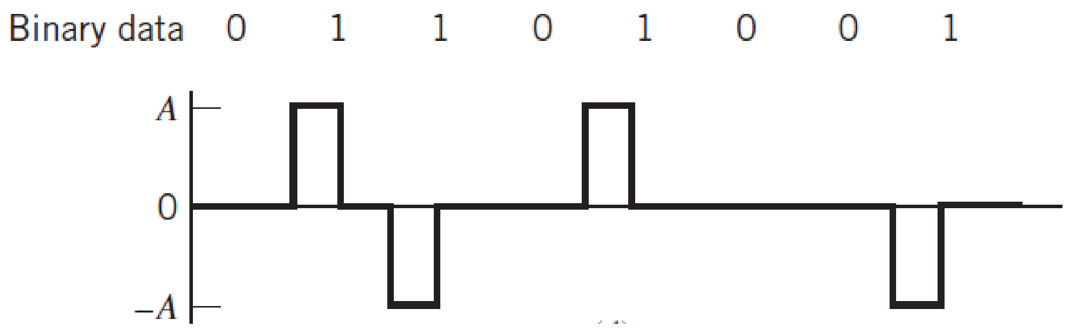


Symbol width

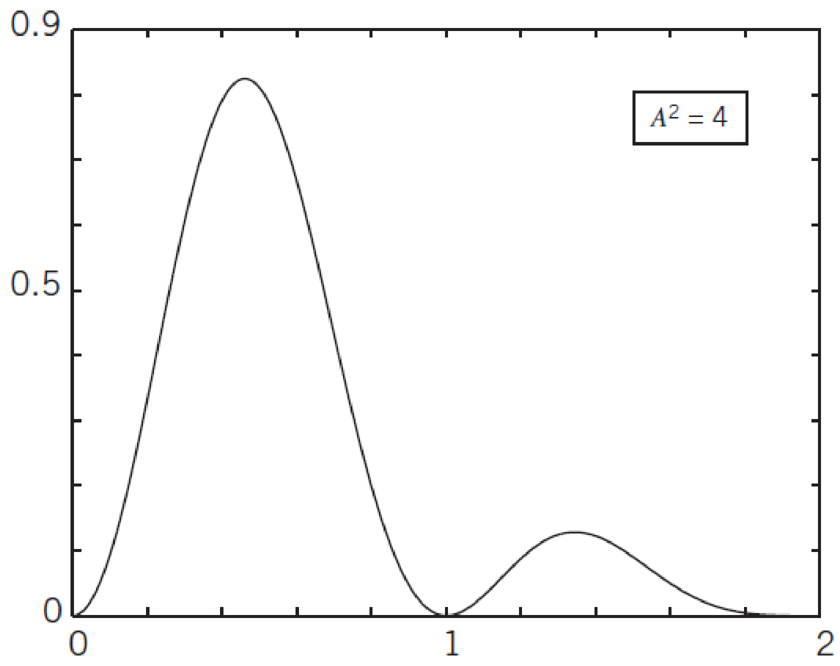
Advantages:
- Delta functions in power spectrum which can be used for bit-timing recovery at receiver

Disadvantages:
- Delta functions consume power

4. Bipolar Return-to-Zero (RZ)



Power Spectrum

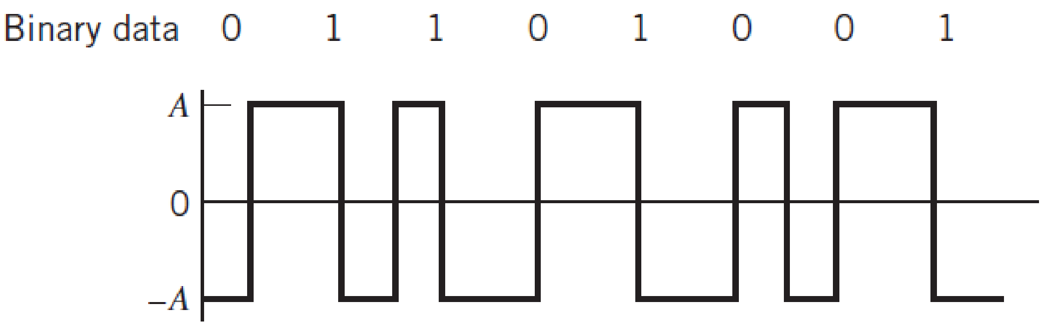


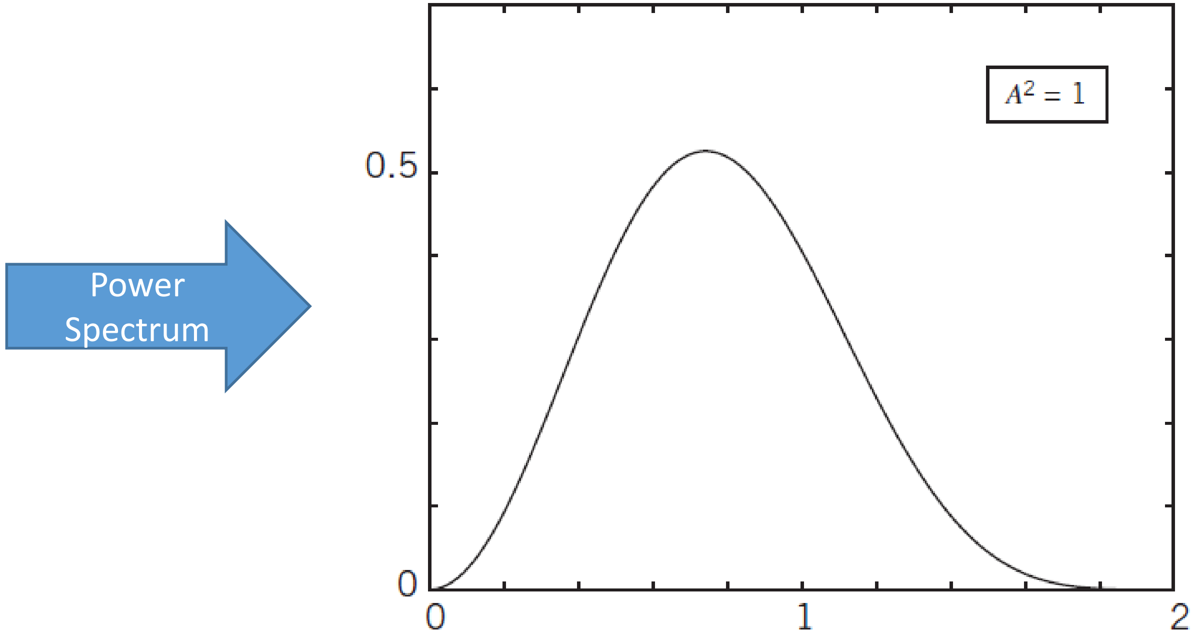
0 = no pulse
1 = positive and negative pulses of equal amplitude (i.e., +A and –A) are used alternately

Advantages:
- No DC (saves power)

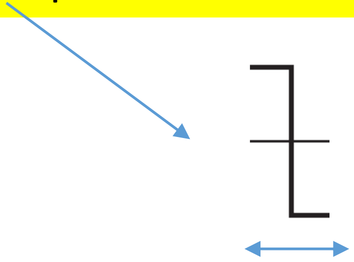
Disadvantages:
- Low frequencies disappear only if 1s and 0s appear with roughly equal probability

5. Manchester Code





0 = pulse whose first half is –A and second half is +A
1 = pulse whose first half is +A and second half is –A



Advantages:
- No DC or low frequencies regardless of the statistics of occurrence of 0s and 1s

Questions?? Thoughts??

EE 322
Digital Communications

with

Dr. Naveed R. Butt

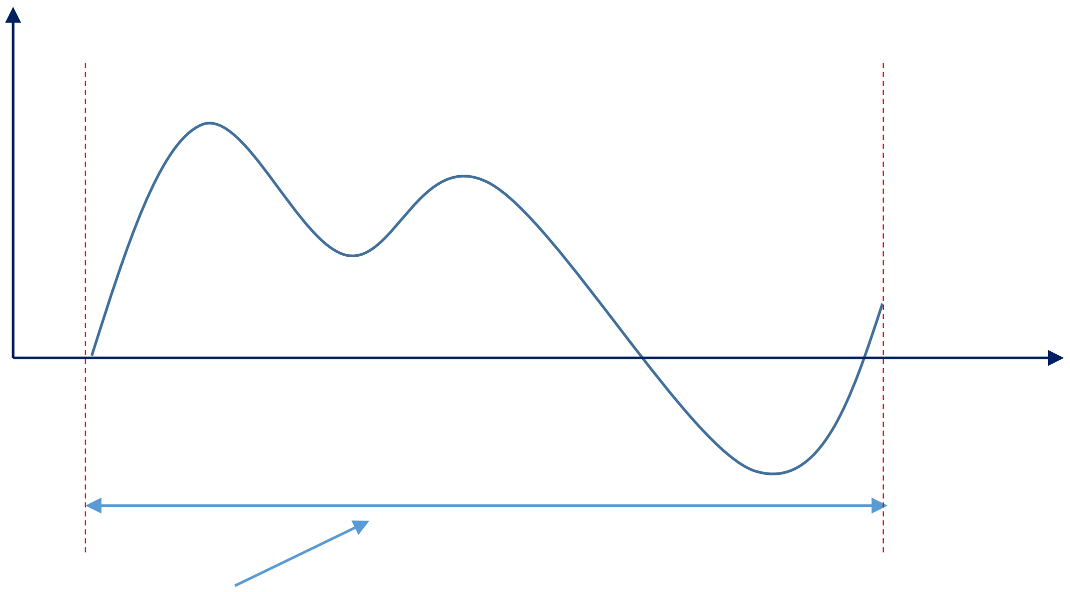
@

Jouf University

Analog → Digital

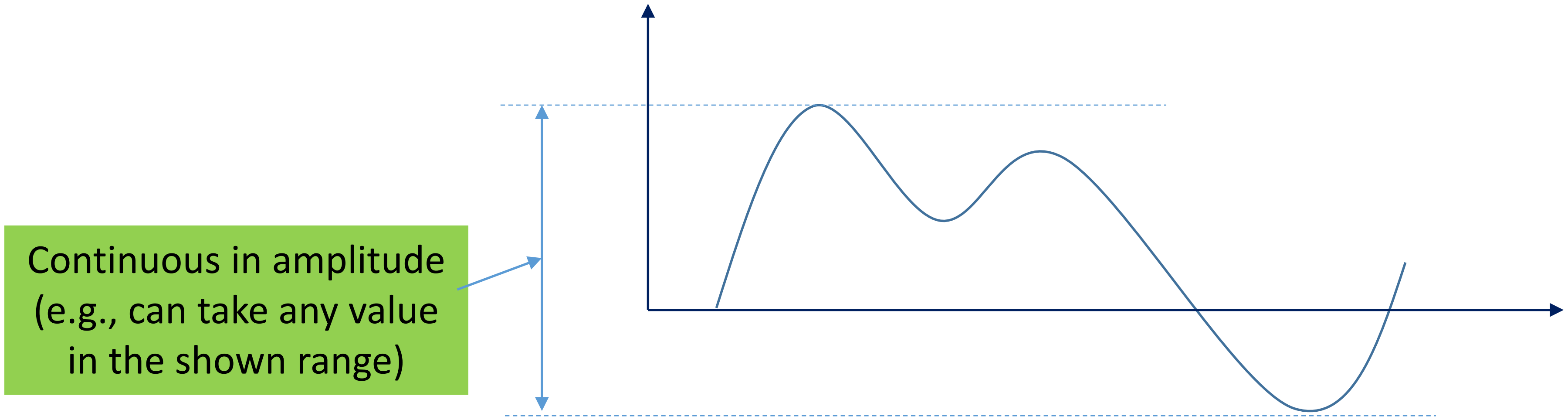
- Most processes in real life are analog.
  - Analog = continuous in time and continuous in magnitude
  - Continuous? No breaks. Can take **any** value in a given range.
  - e.g., temperature in this room

Classroom Temperature



Continuous in time (e.g., can take any value in the shown range)

Classroom Temperature



Analog → Digital

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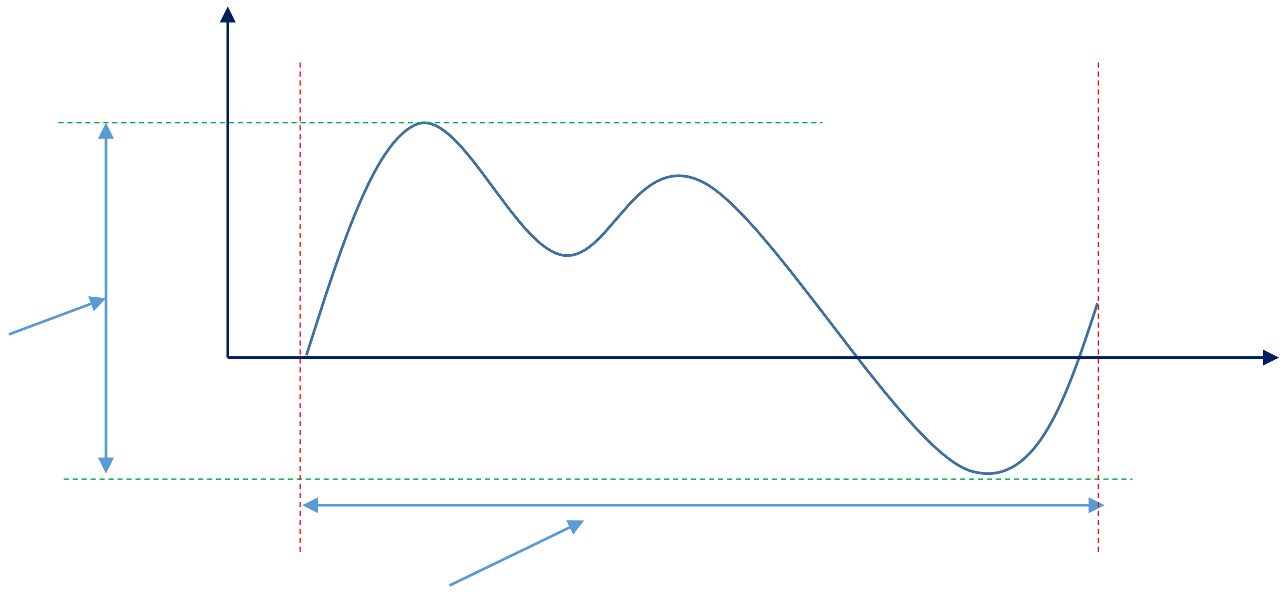
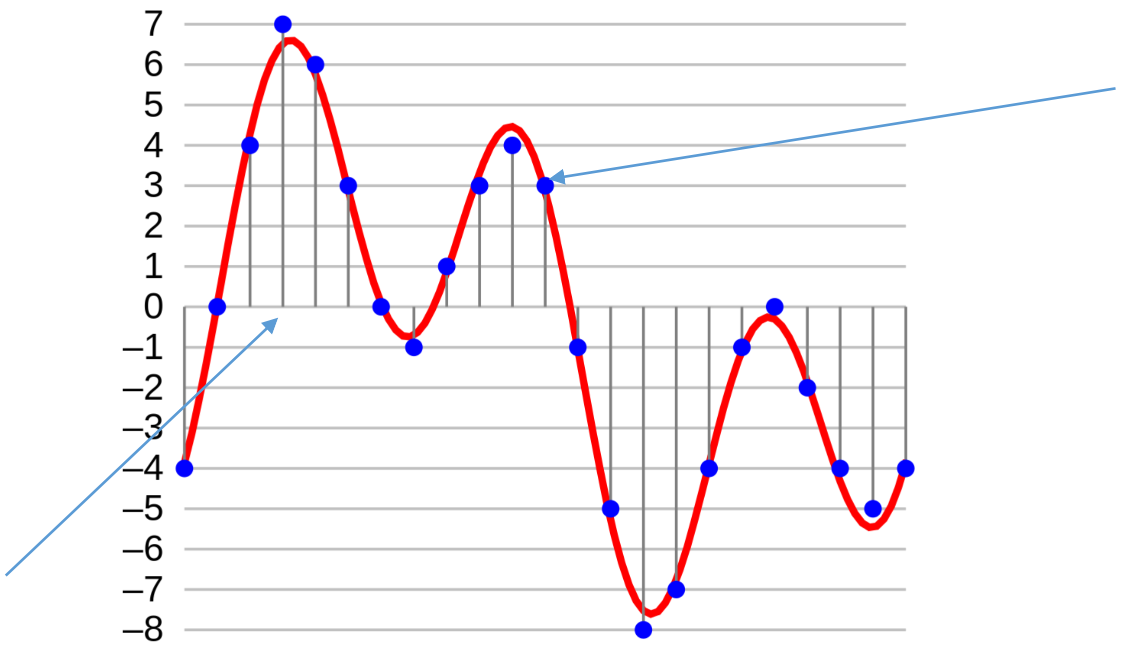


Analog → Digital

- Digital Communications works with digital signals
  - Digital = discontinuous in time and “discontinuous” in magnitude

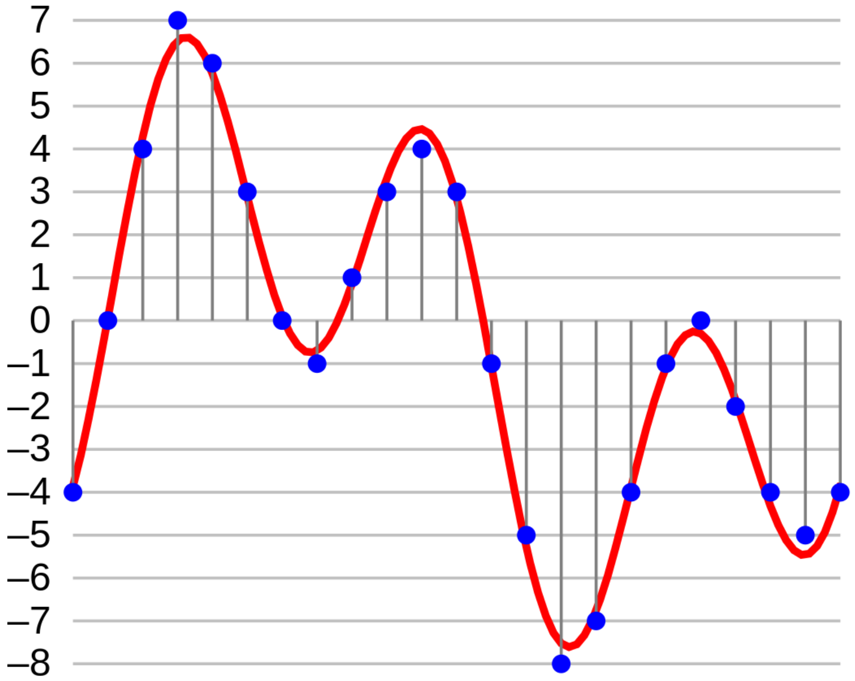


discontinuous in time (e.g., values taken only at fixed times but not in-between)

“discontinuous” in amplitude (e.g., can not take values between 3 and 4, like 3.5 etc.)

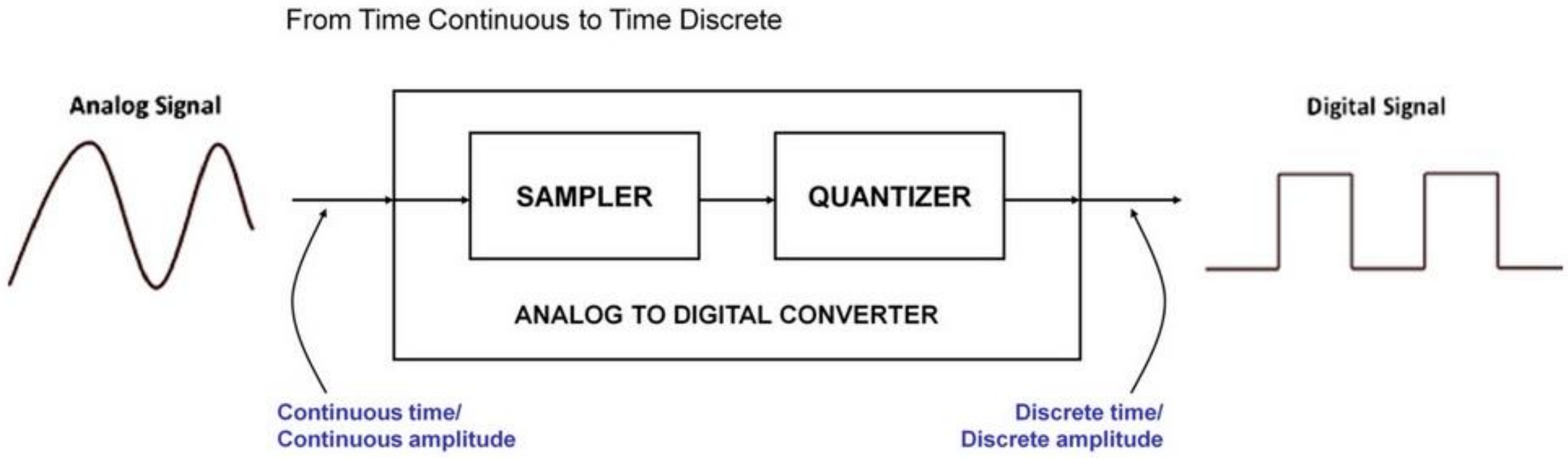
Analog → Digital

- Where do digital signals come from?
  - In most cases, we take analog signals and convert them into digital ones!
  - e.g., measure room's temperature every two seconds (0, 2, 4, 6 …) , and round off measurements to fixed values (0, 1, 2, 3, 4 …)

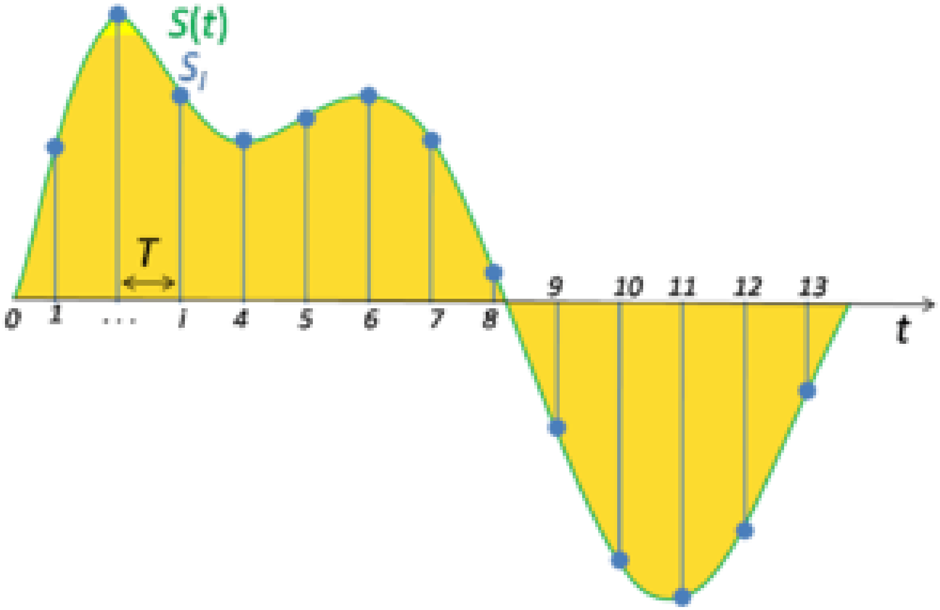


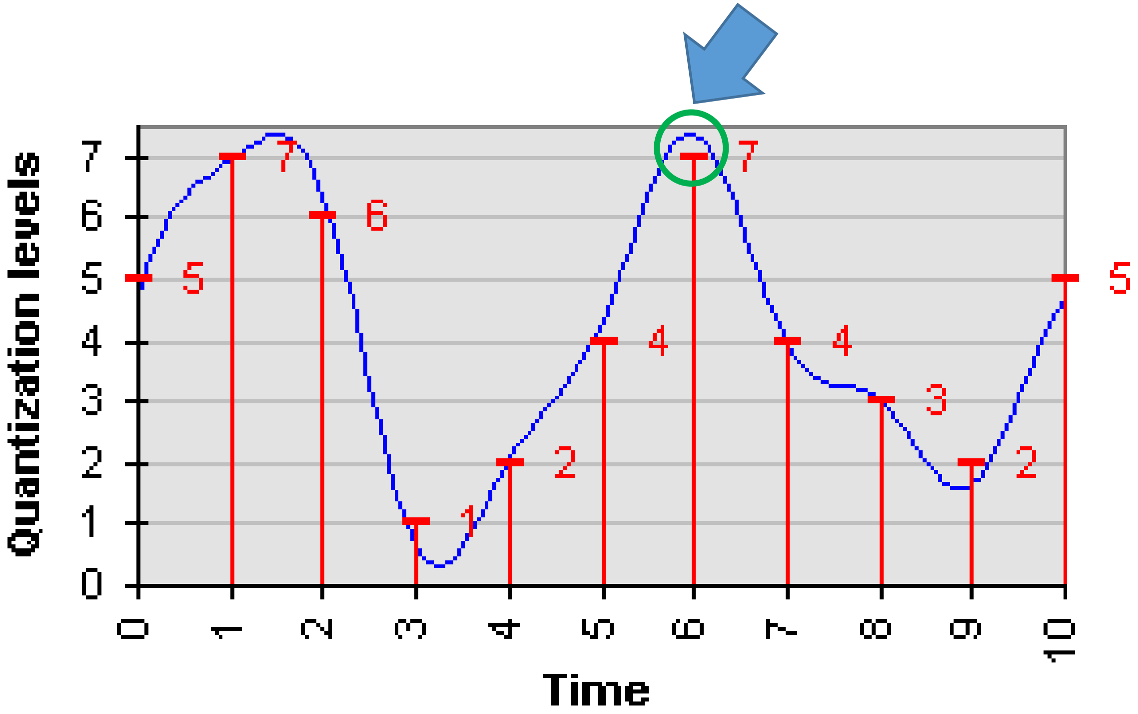
Analog → Digital

- Converting analog signals to digital is a **two step process**
  - Step 1: record signal only at fixed time intervals (called ***Sampling***)
  - Step 2: “round-off” recorded samples to fixed set of values (called **Quantization**)

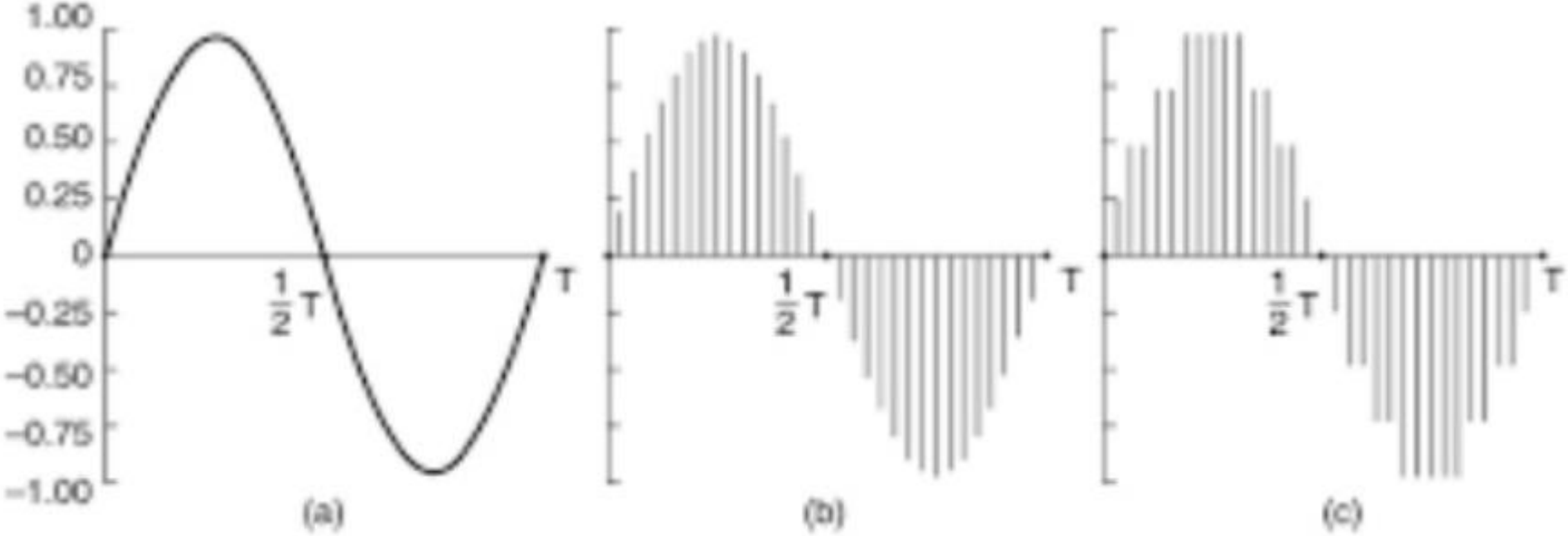


Sampling & Quantization

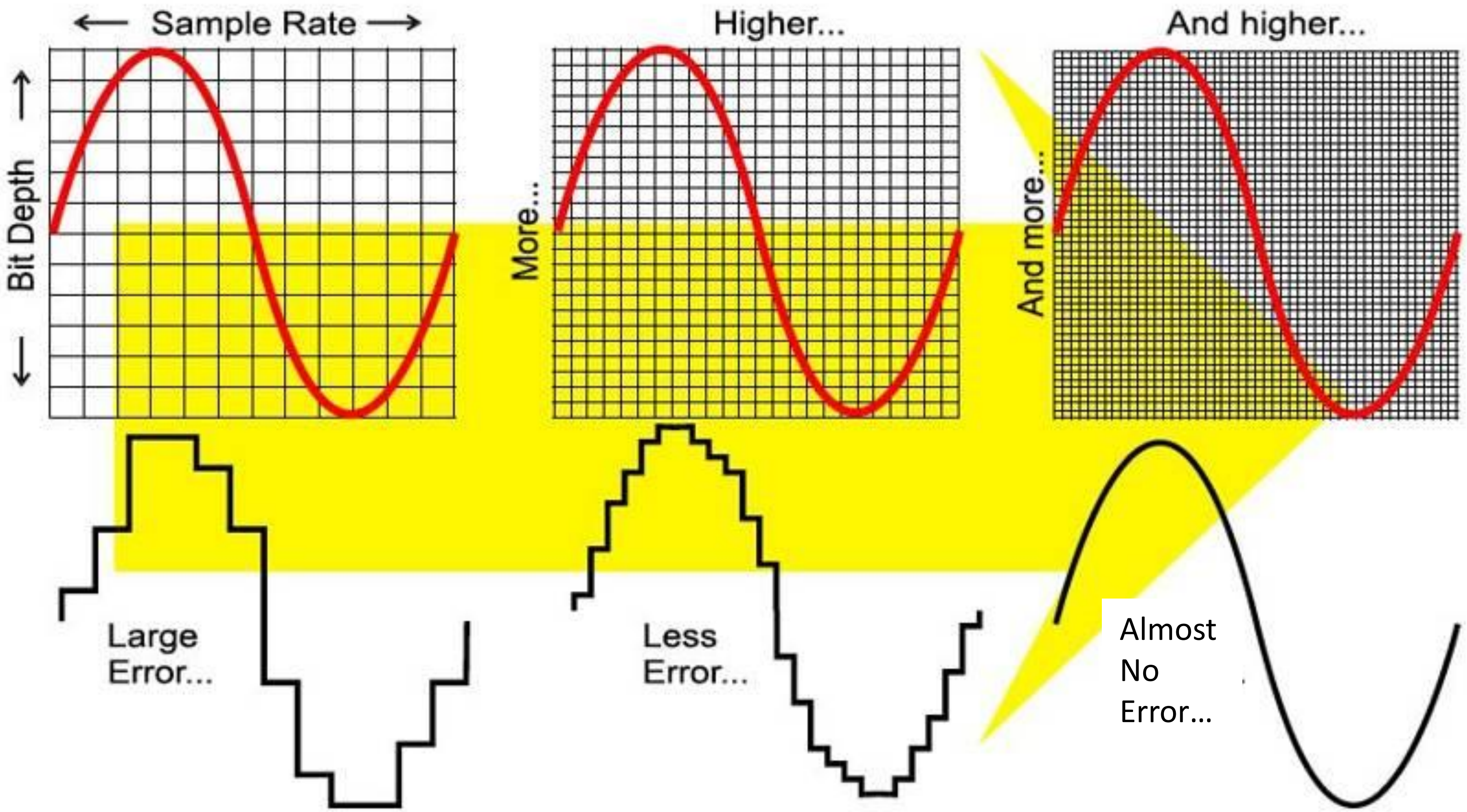




Sampling & Quantization

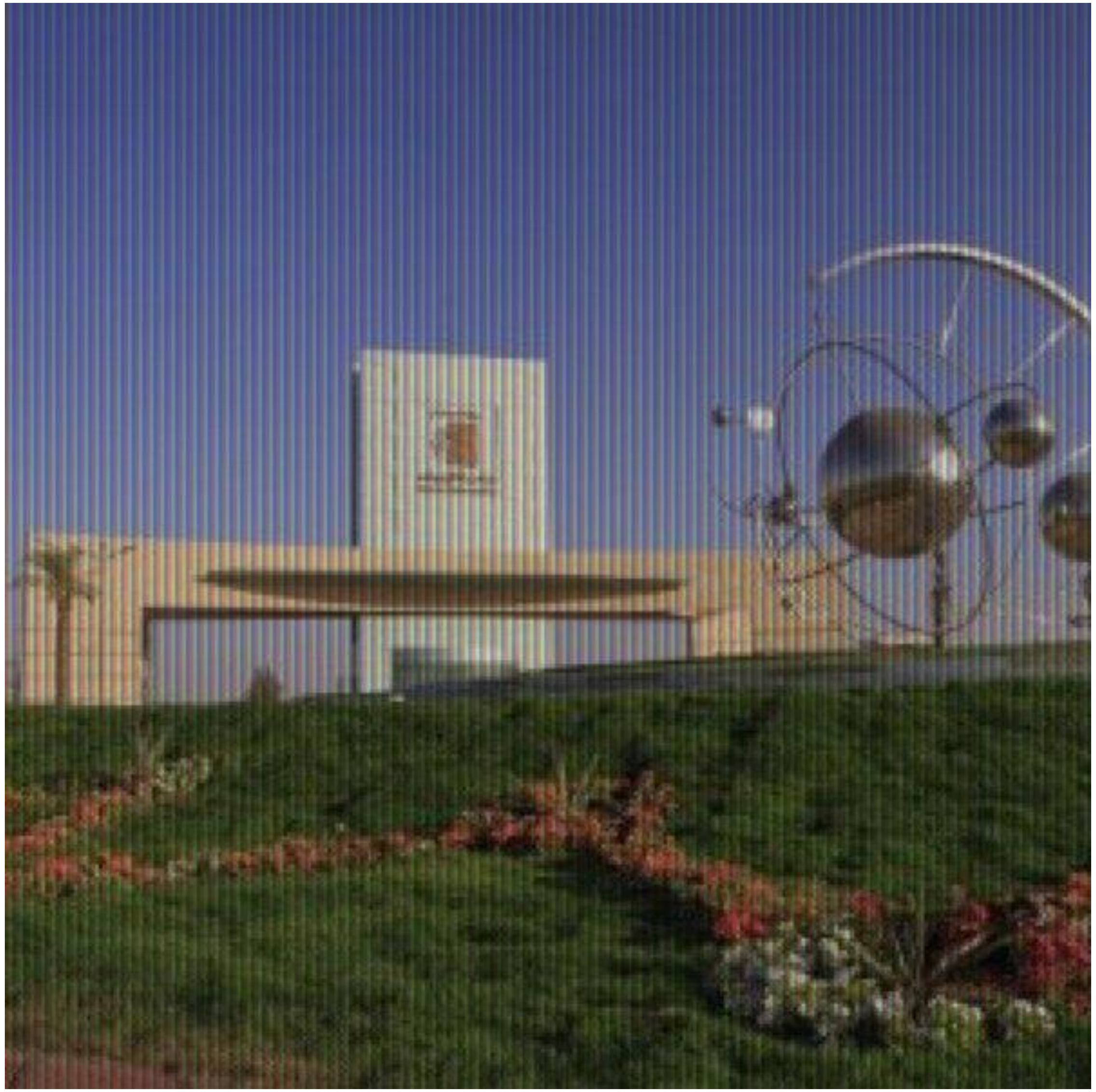


Good Sampling & Quantization



Analog → Digital : Important Requirement

- The information content of the signal should not change!
  - (or change as little as possible)

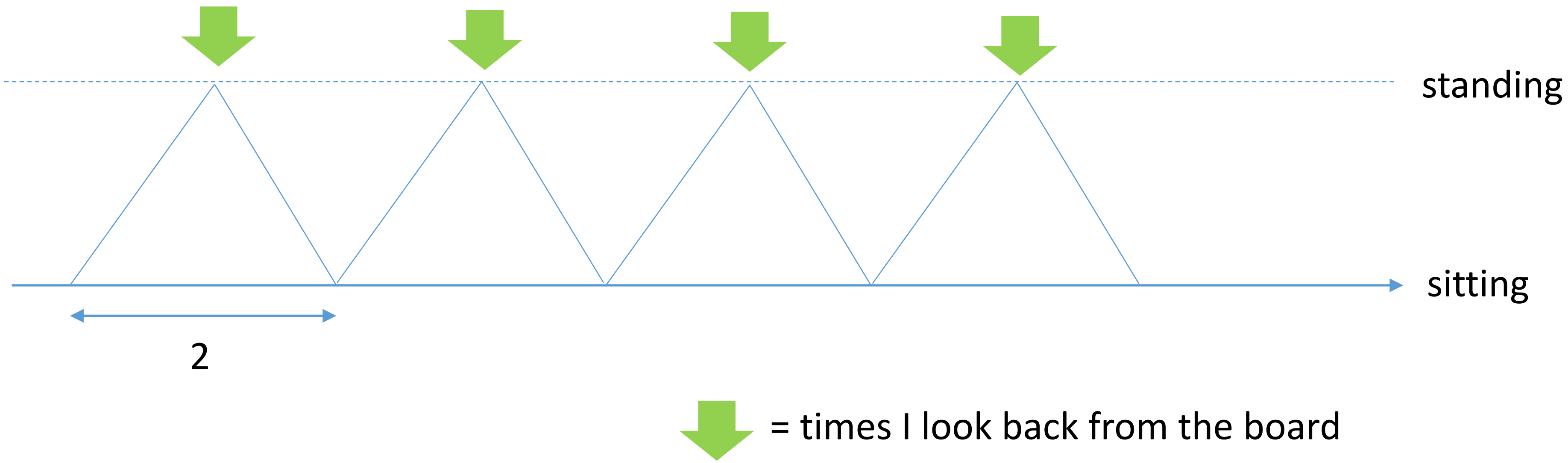


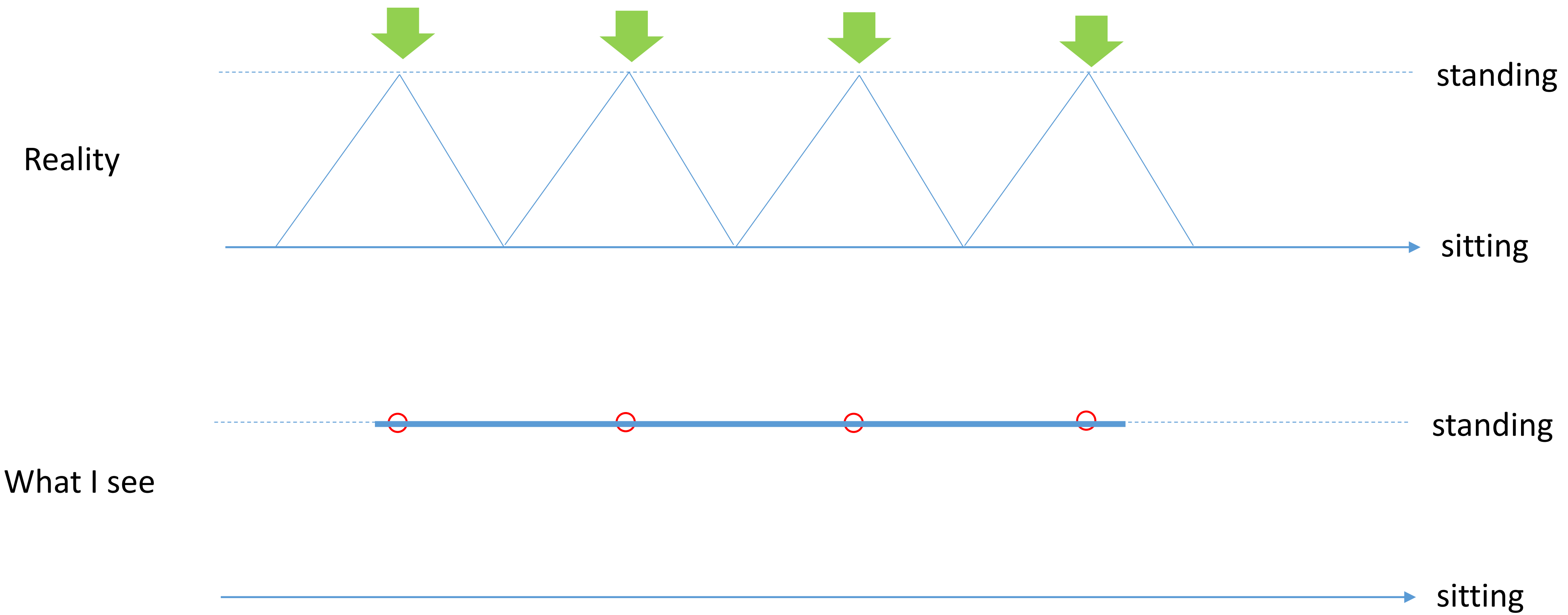

Q. How many samples are enough?

- When we sample a signal, we want the sampled signal to have the same information content (e.g., frequencies) as the original analog signal
- Then how many samples are enough to make sure of this?

Student sit-ups!!

Imagine a bored student starts doing sit-ups in the class. Let's say he takes two seconds to complete each sit-up. If I look back from the board every two seconds, what may I see?

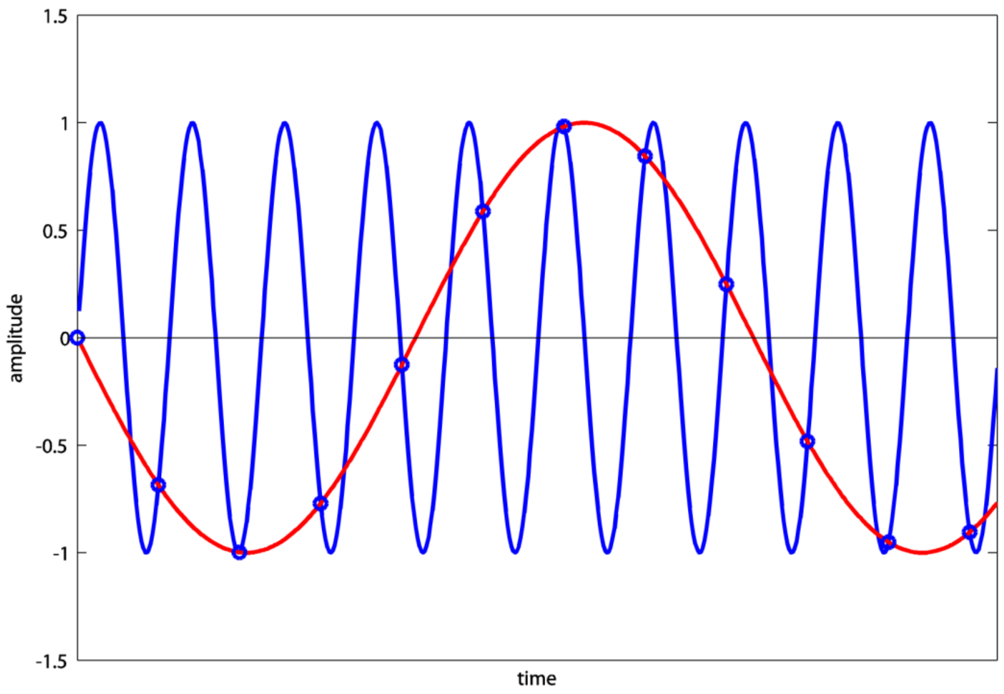




- I always find the student in one position (e.g., standing)
- A high frequency event appears to me as a low frequency event (zero frequency in this case)
- In order to "catch" the student in different positions I should look back perhaps twice in two seconds!!

Aliasing = when a **high frequency looks like a low frequency** due to insufficient sampling

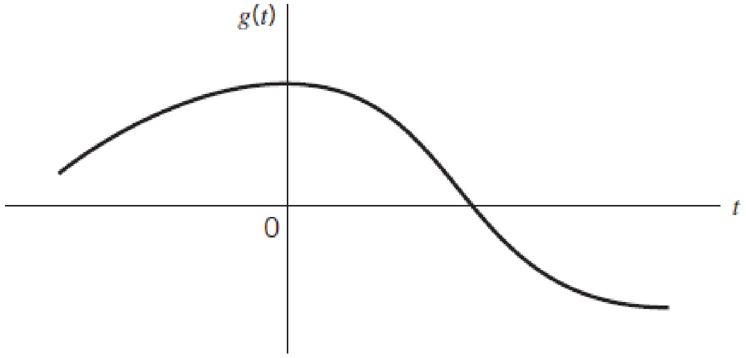
- In the sit-ups example, the high frequency appeared to me as a low (zero) frequency because of insufficient (too slow) sampling.
  - This is called "aliasing"



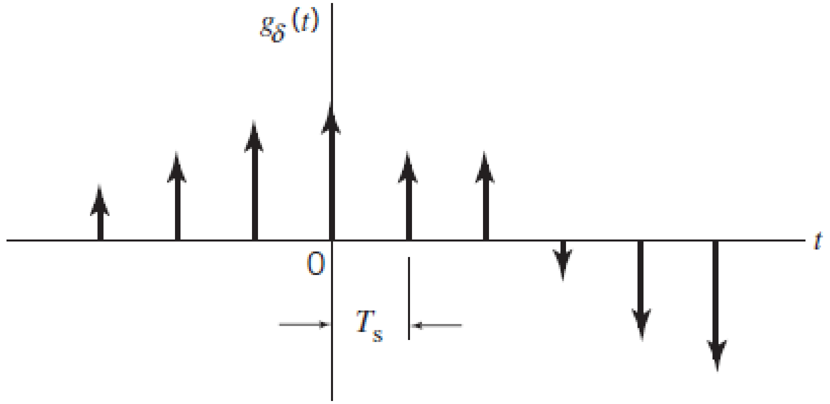
Q. How many samples are enough?

- Student sit-ups example
  - We see from the example that insufficient sampling can change the information in the signal
  - Also, it seems that **sampling at least twice** the signal frequency can help
  - “Sample twice the highest signal frequency” is actually an important result in signal processing, called Nyquist Rate.
  - Next, we derive the Nyquist Rate formally (mathematically)

How can we write a sampled signal mathematically?



We create a sampled signal from $g(t)$ by measuring it every T_s seconds.



Measured samples $g(nT_s)$

Mathematical formulation of the measured samples

$$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\,\delta(t - nT_s)$$

Unit impulse

$$\delta(t - nT_s) = \begin{cases} 1 & t = nT_s \\ 0 & t \neq nT_s \end{cases}$$

Information (e.g., frequency) content of a signal

Original signal	Frequency content of the original signal
$g(t)$	$G(f)$
$g_\delta(t) = \sum_{n=-\infty}^{\infty} g(nT_s)\delta(t - nT_s)$	$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - mf_s)$
Sampled signal	Frequency content of the sampled signal

Ideally we want: $G_\delta(f) = G(f)$

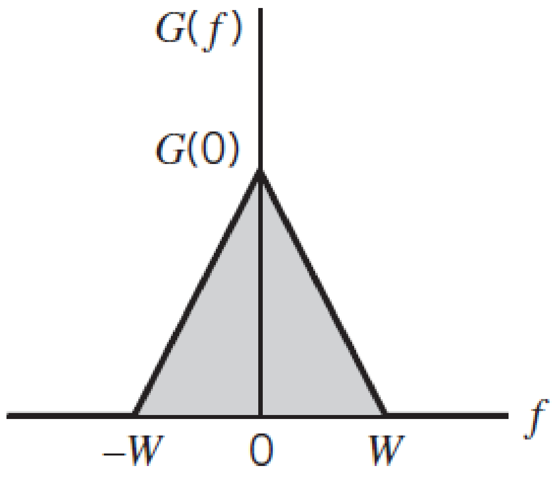
i.e., we want frequency content of sampled signal to be the same as that of the original signal

$$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - m f_s)$$

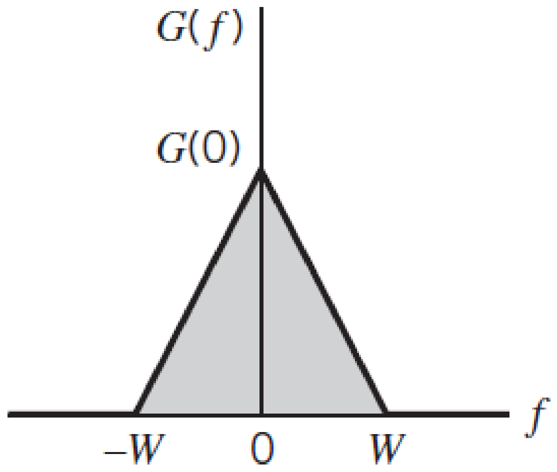
What do these extra terms mean?

For a “bandlimited” signal …

- Let us consider a bandlimited signal with frequency content shown below
  - The signal has **bandwidth** 2W and satisfies $G(f) = 0 \text{ for } |f| \geq W.$



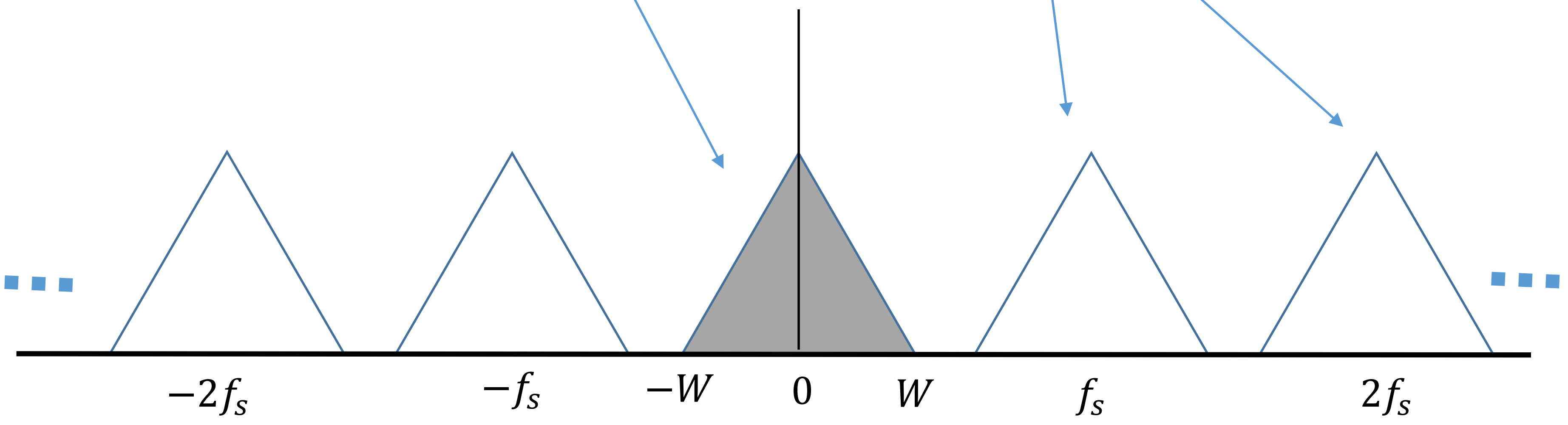
Means that the signal has no frequencies higher than ***W***



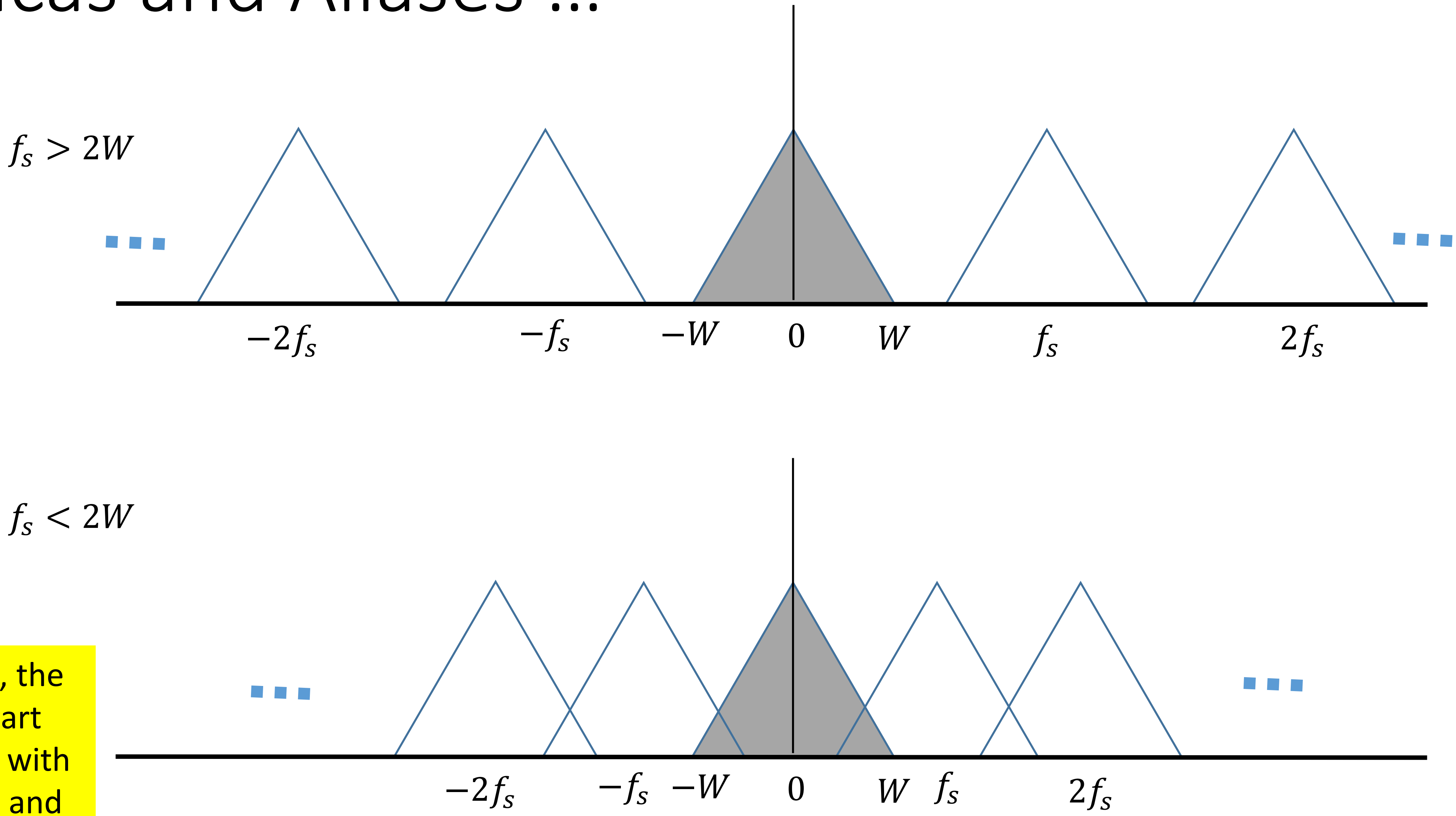
Original signal spectrum

Copies (called **Replicas**) of the original signal spectrum centered at $\pm f_s, \pm 2f_s, \ldots$

$$G_\delta(f) = f_s G(f) + f_s \sum_{\substack{m=-\infty \\ m \neq 0}}^{\infty} G(f - m f_s)$$



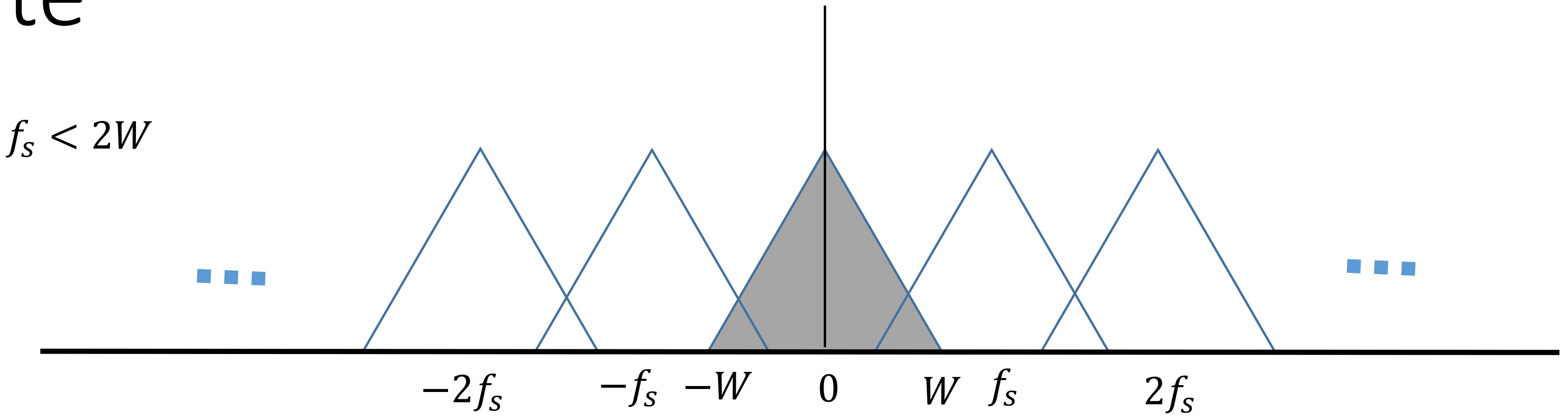
Replicas and Aliases …



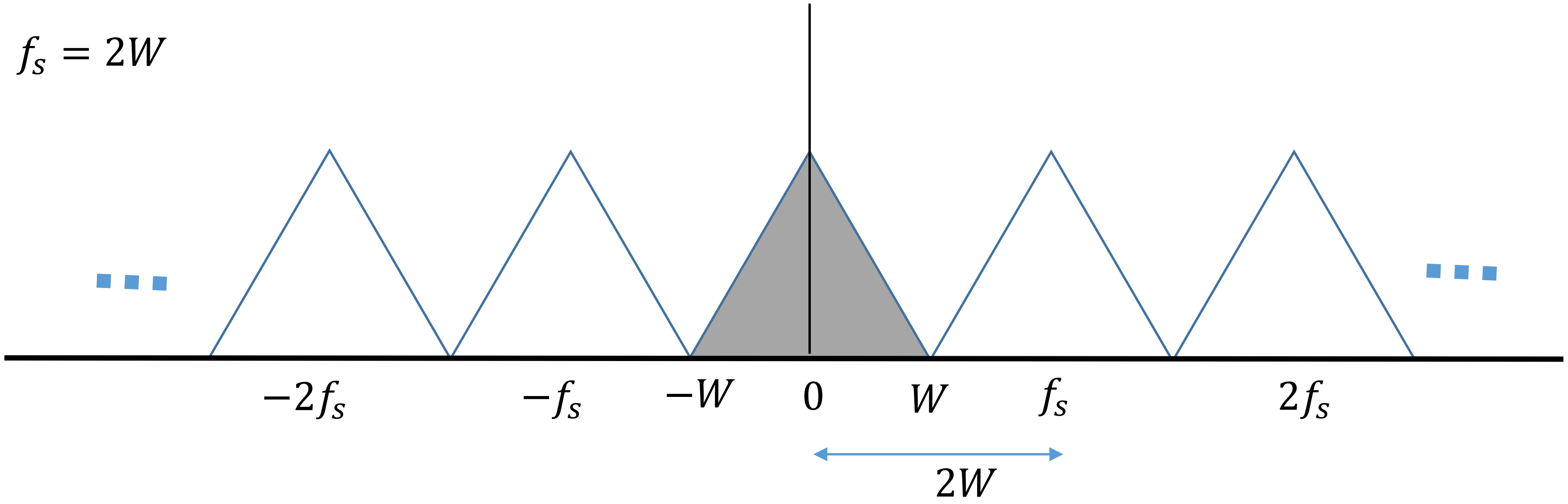
If $f_s < 2W$, the replicas start overlapping with the original and cause "aliasing"

Nyquist Rate

If $f_s < 2W$, the replicas start overlapping with the original and cause “aliasing”



To avoid that, and to keep the original signal content unchanged, we must sample at at least twice the highest signal frequency. This is called the **Nyquist Rate**.



So,

- If the signal is bandlimited and sampling frequency is set to at least twice the highest frequency of the signal then we can easily recover the original signal from the sampled signal

1. $G(f) = 0$ for $|f| \geq W$.
2. $f_s = 2W$.

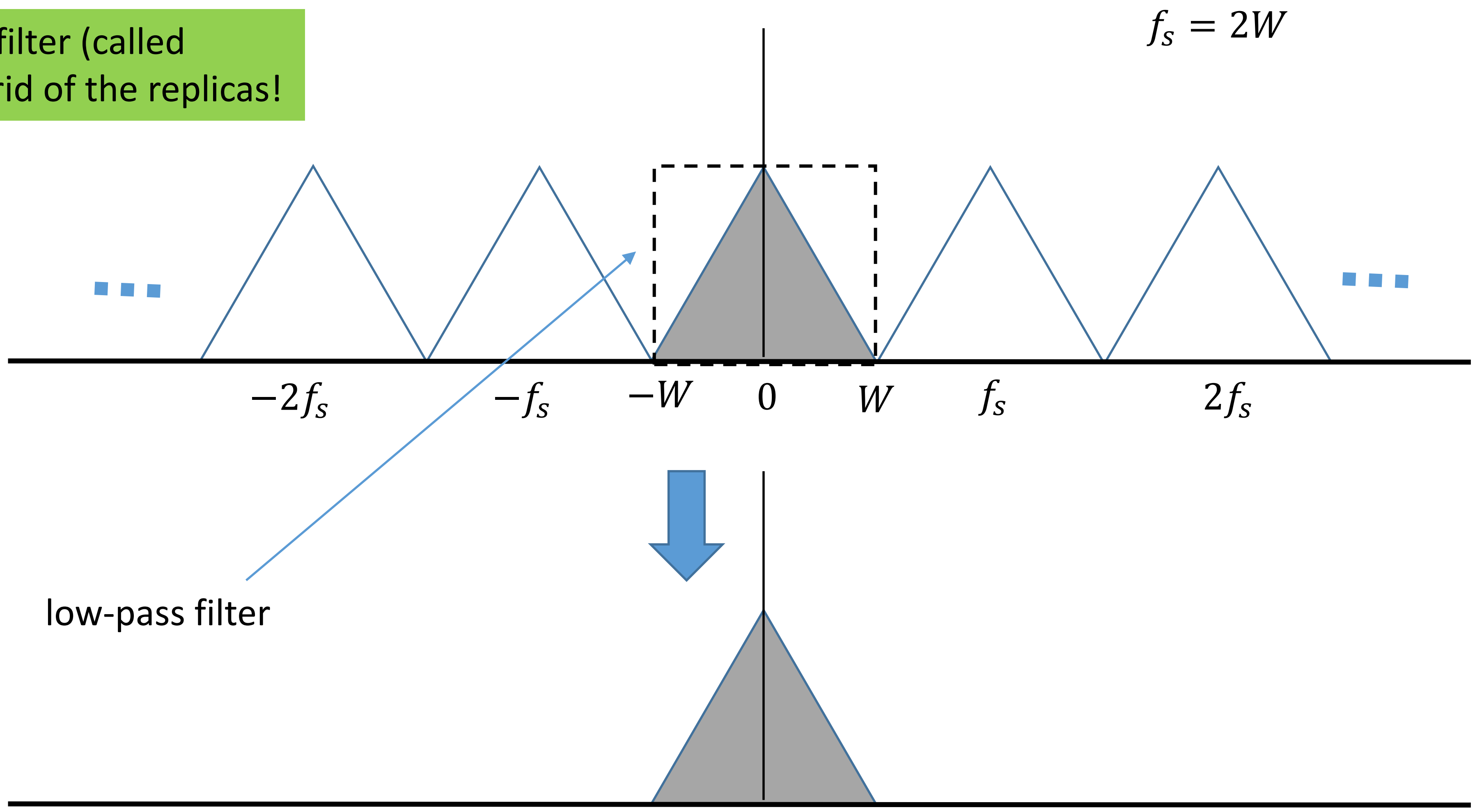
$\Rightarrow$

$$G(f) = \frac{1}{2W} G_\delta(f), \qquad -W < f < W$$

And how do we get rid of the replicas?

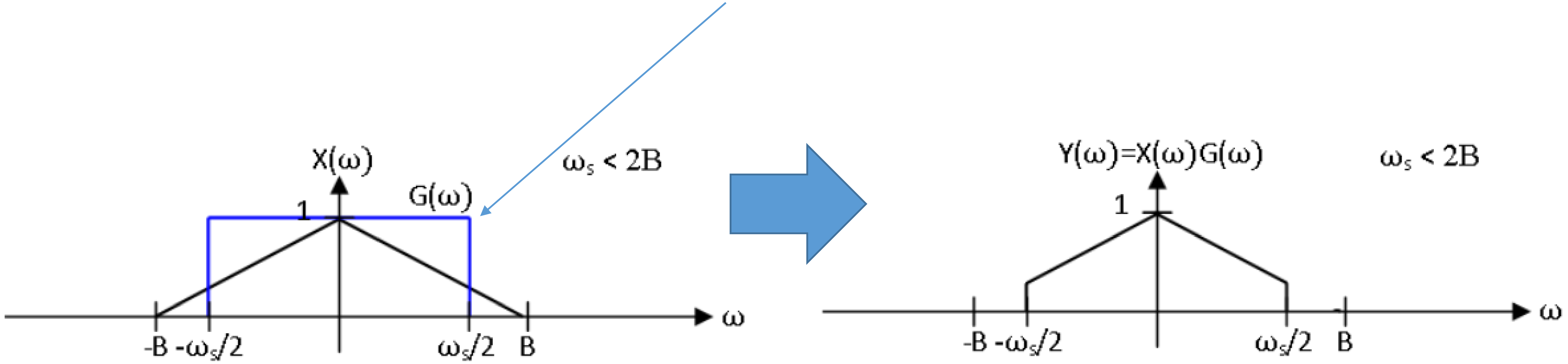
We can use a low-pass filter (called **Reconstruction Filter**) to get rid of the replicas!

Note: not possible if $f_s < 2W$

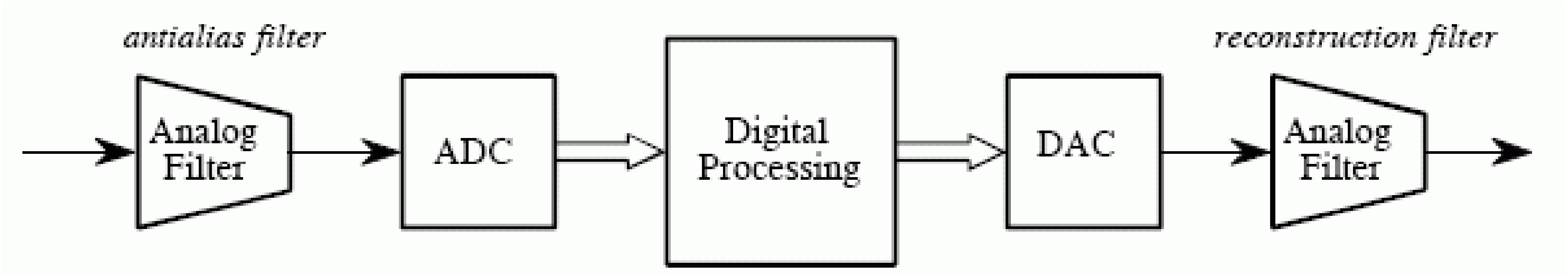


And what if the signal is not bandlimited?

- Most signals of interest are bandlimited
  - e.g., human voice
- We can mostly define a range of frequencies we are interested in and remove the higher ones *before* sampling
  - This process is done through an **Anti-aliasing Filter**



Analog → Digital → Analog

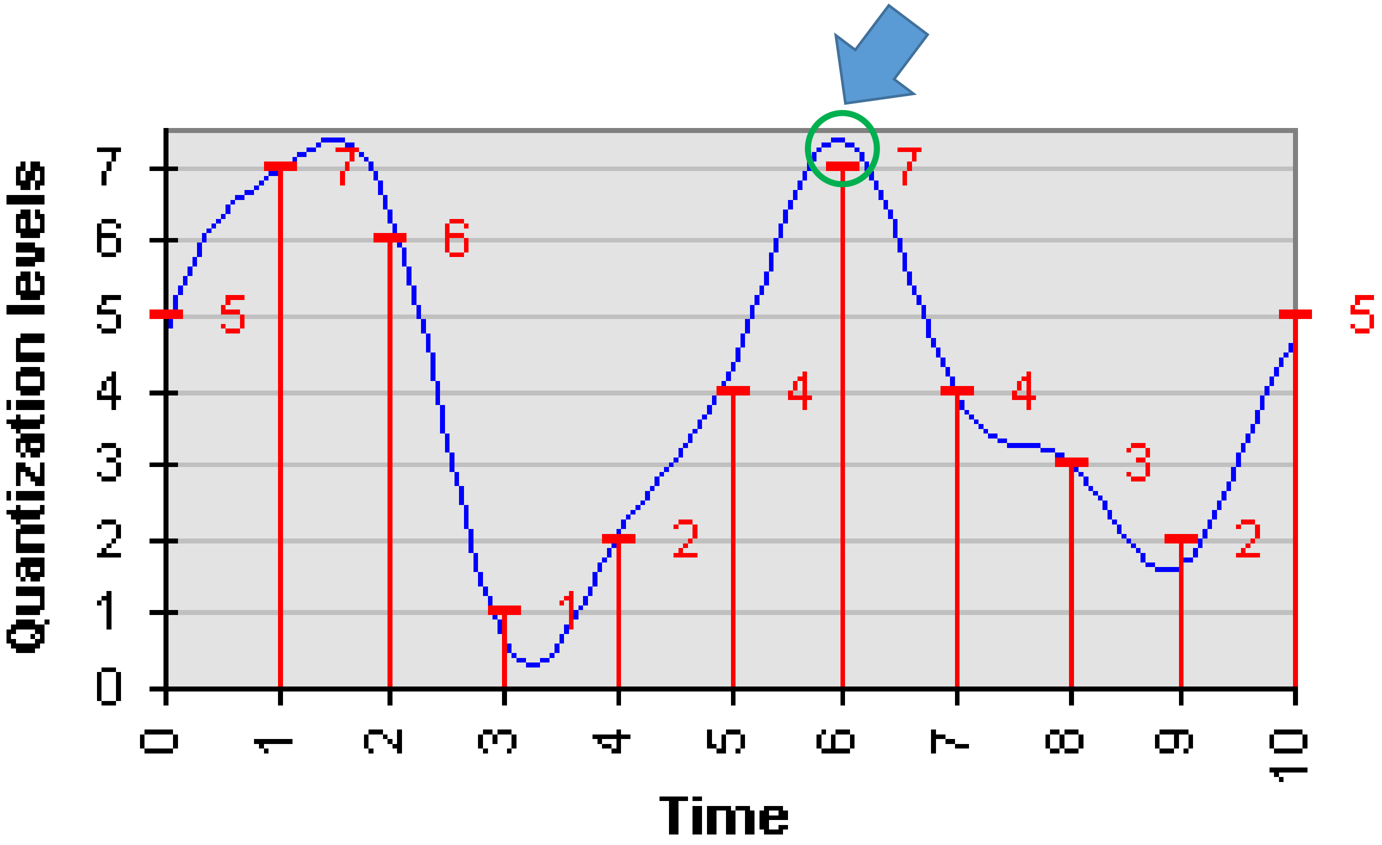


What aliasing looks like

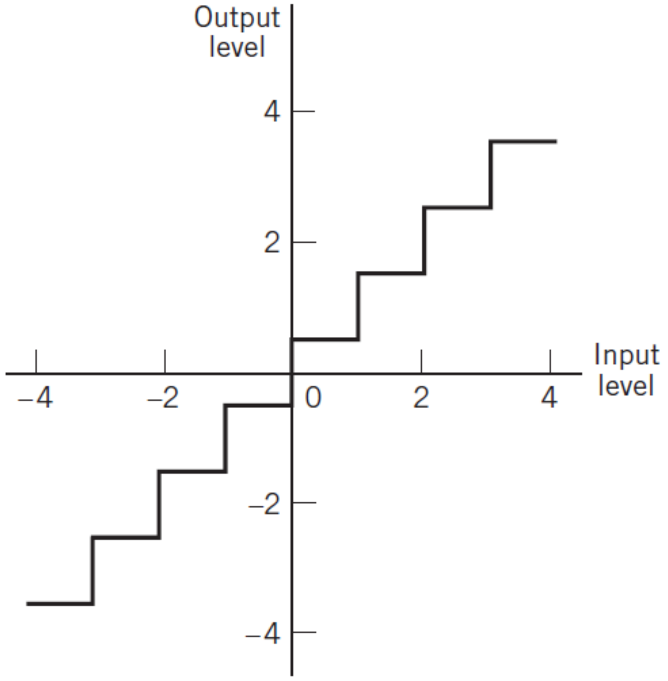
Sampling Summary

- Sampling Theorem
  - A bandlimited signal with no frequency above W can be fully recovered from its samples if sampled at $2W$ (or faster)
  - A bandlimited signal with no frequency above W is fully described by its samples if sampled at $2W$ (or faster)
- Ensuring correct sampling and reconstruction
  - Apply anti-aliasing filter before sampling
    - To avoid aliasing
  - Apply low-pass filter for reconstruction
    - To get rid of replicas in reconstructed signal

Quantization and Quantization Noise



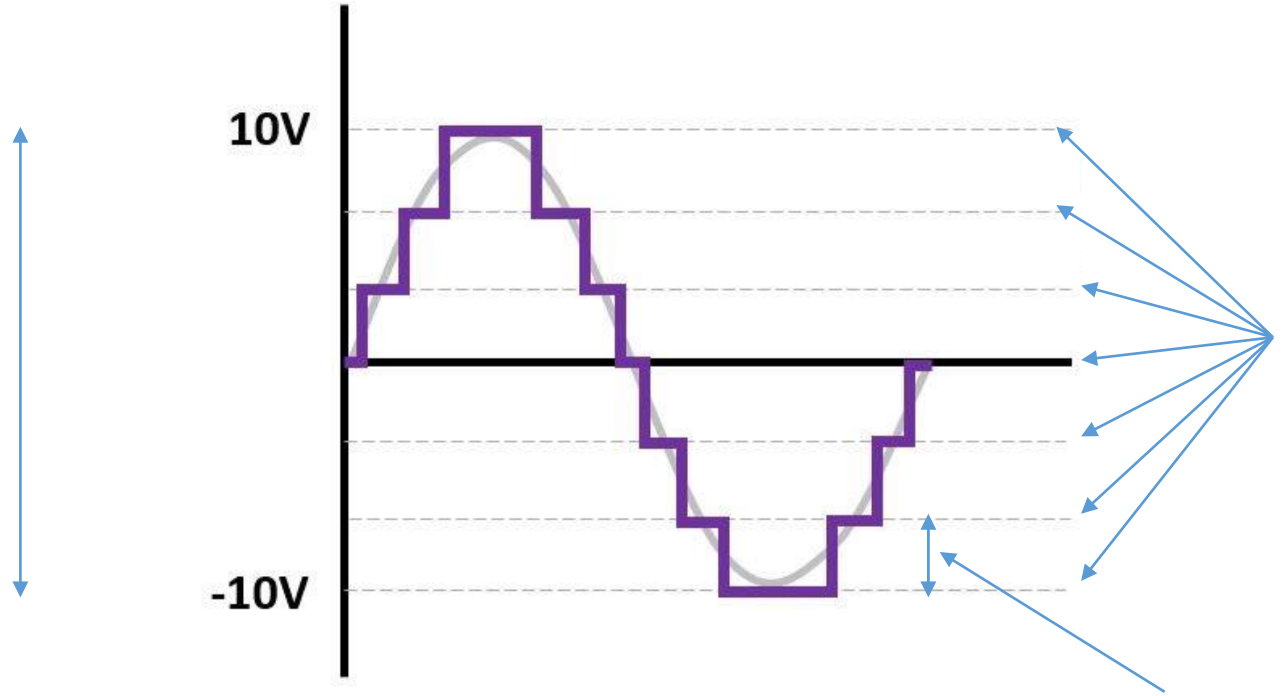
Uniform Quantizer



Input-output graph of a uniform quantizer

Signal Range, Number of Quantization Levels (L) and Step-Size (Δ)

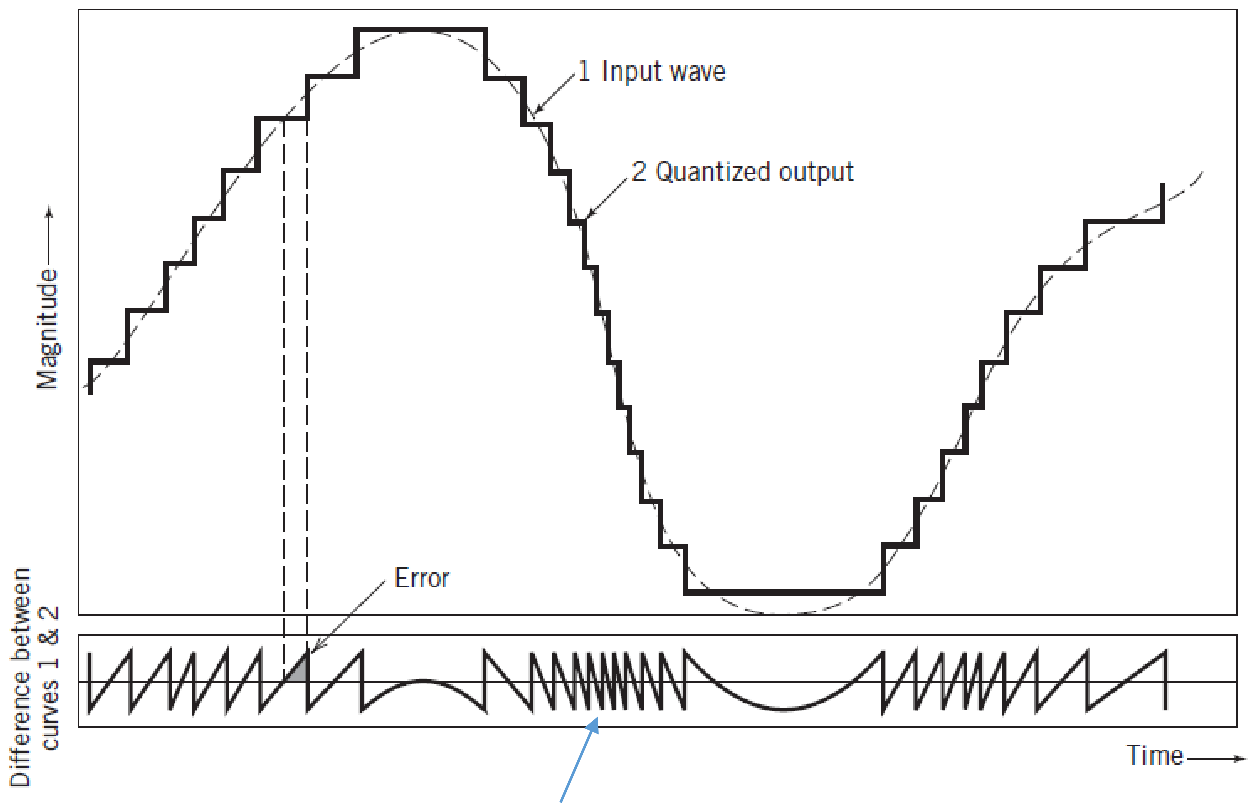
Signal Range = $\mathbf{2m_{max}}$
($m_{max} = 10\ V$)



$L = 7$
(seven quantization levels)

Step-Size $\Delta = \frac{2m_{\max}}{L}$

Quantization Noise



Quantization Noise ($\boldsymbol{Q}$)

Quantization and Quantization Noise

- Just like sampling, **we do not want quantization to affect the information content of the signal**
- Unfortunately, that is not always possible
  - Quantization introduces some amount of error in the signal called **Quantization error** or **Quantization Noise**
- We can try to **keep the error/noise as low as possible**
  - For that we must first study the properties of the quantization noise

Statistical Properties of Quantization Noise

- Assumptions/Definitions
  - Uniform Quantization (quantization levels are equally spaced)
  - Number of quantization levels = $\boldsymbol{L}$
  - Original signal amplitude lies in the range $[-\boldsymbol{m_{max}}\ ,\ \boldsymbol{m_{max}}]$
  - Quantization Noise $\boldsymbol{Q}$ = amplitude difference between original and quantized signal
- With the above assumptions/definitions, we describe distribution, mean, and variance of the quantization noise $\boldsymbol{Q}$

Distribution of Quantization Noise Q

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \le \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

The Quantization Noise of a uniform quantizer is **uniformly distributed**

$$\Delta = \frac{2m_{\max}}{L}$$

quantization step-size

Mean of Quantization Noise Q

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \le \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta = \frac{2m_{\max}}{L}$$

$$\mu_Q = \mathbb{E}[Q] = 0$$

The Quantization Noise of a uniform quantizer has **zero mean**.

Variance of Quantization Noise Q

$$f_Q(q) = \begin{cases} \frac{1}{\Delta}, & -\frac{\Delta}{2} < q \le \frac{\Delta}{2} \\ 0, & \text{otherwise} \end{cases}$$

$$\Delta = \frac{2m_{\max}}{L}$$

$$\sigma_Q^2 = \mathbb{E}[Q^2]$$

$$= \int_{-\Delta/2}^{\Delta/2} q^2 f_Q(q)\, \mathrm{d}q$$

$$= \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} q^2\, \mathrm{d}q = \frac{\Delta^2}{12}$$

The Quantization Noise of a uniform quantizer has variance (power) proportional to the **square of the step-size**.

Q. How to reduce quantization noise?
A. Increase L (number of quantization levels)

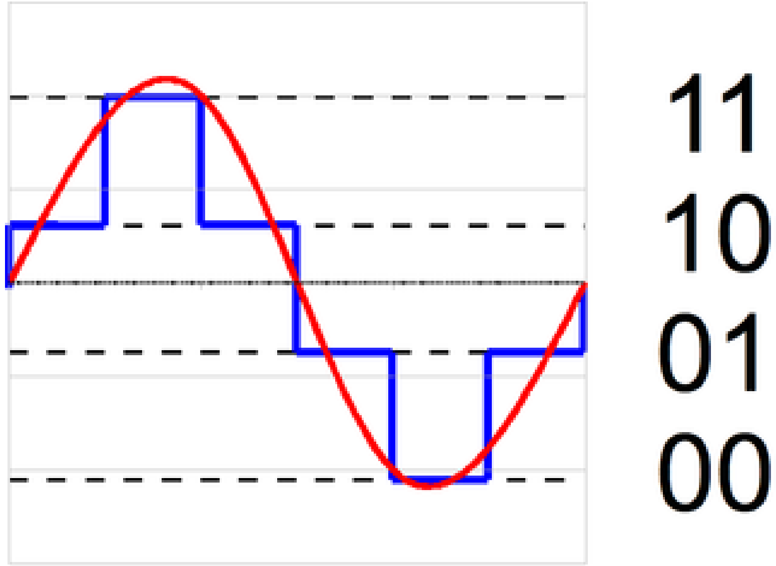
$$\sigma_Q^2 = \mathbb{E}[Q^2] = \frac{\Delta^2}{12}$$

The Quantization Noise of a uniform quantizer has variance (power) proportional to the **square of the step-size**.

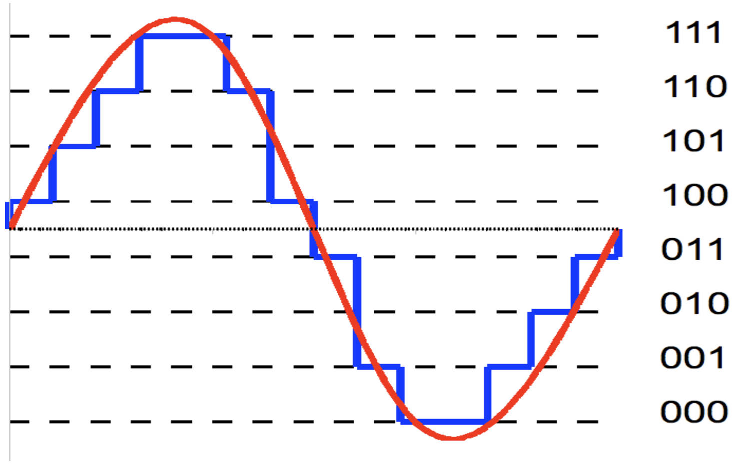
$$\Delta = \frac{2m_{\max}}{L}$$

If the number of quantization levels ($\boldsymbol{L}$) is large, then the step-size ($\boldsymbol{\Delta}$) is small and so is the power of quantization noise

Q. How to reduce quantization noise?
A. Increase L (number of quantization levels)

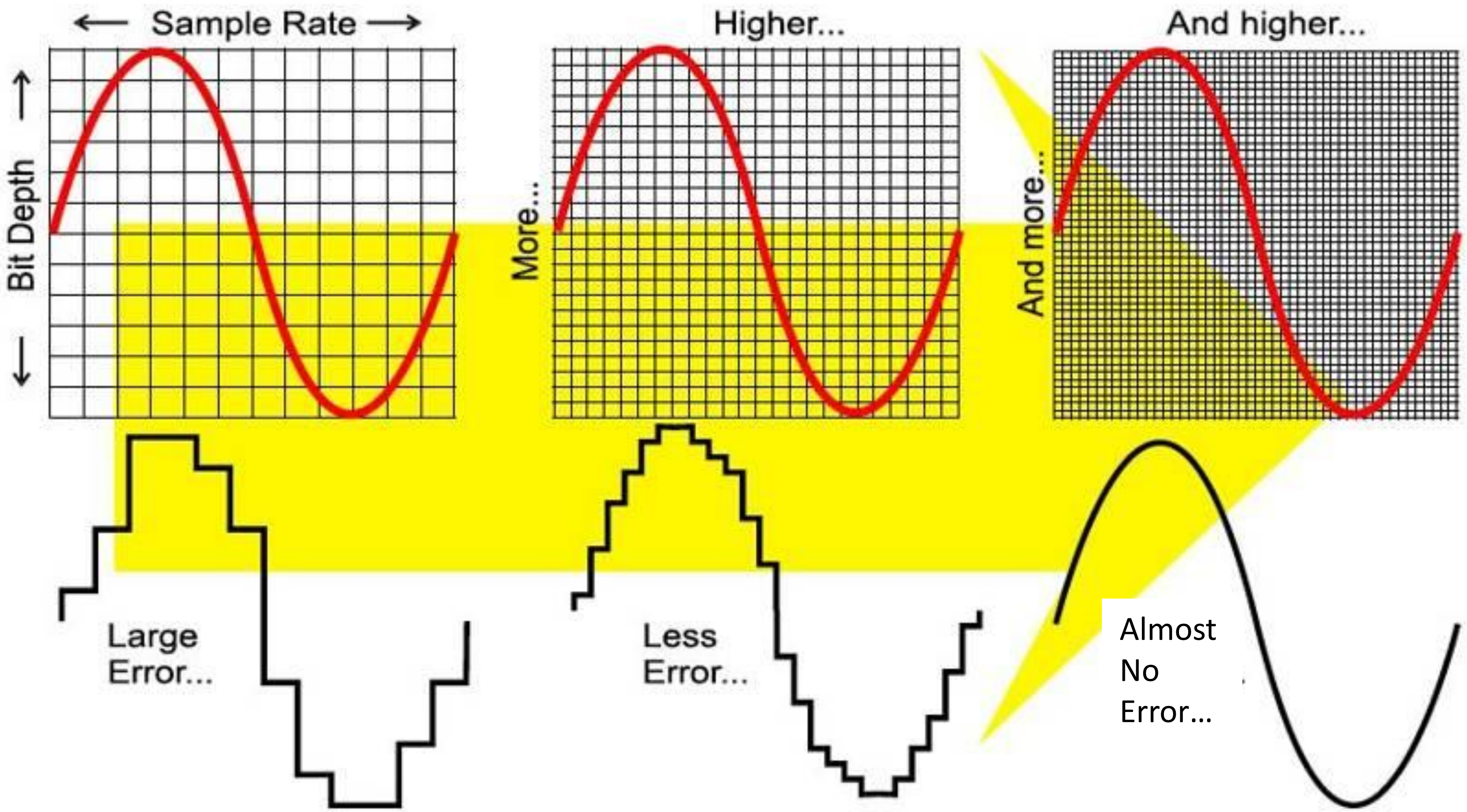


2-bit (L = 4) quantization.
High quantization noise.



3-bit (L = 8) quantization.
Lower quantization noise.

Again, Good Sampling & Quantization



Questions?? Thoughts??

EE 322
Digital Communications

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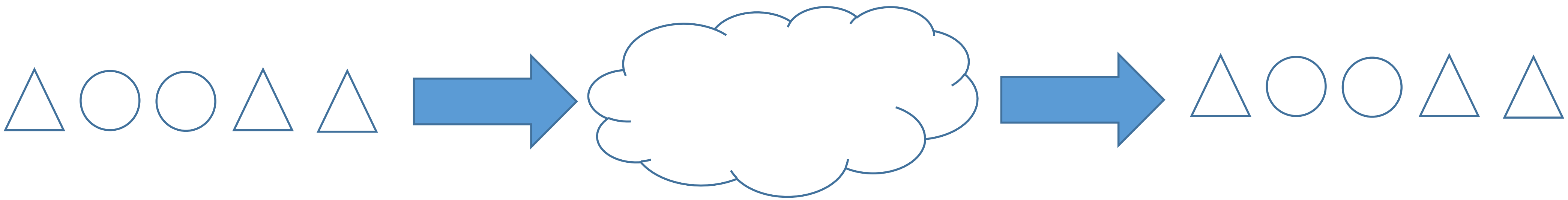
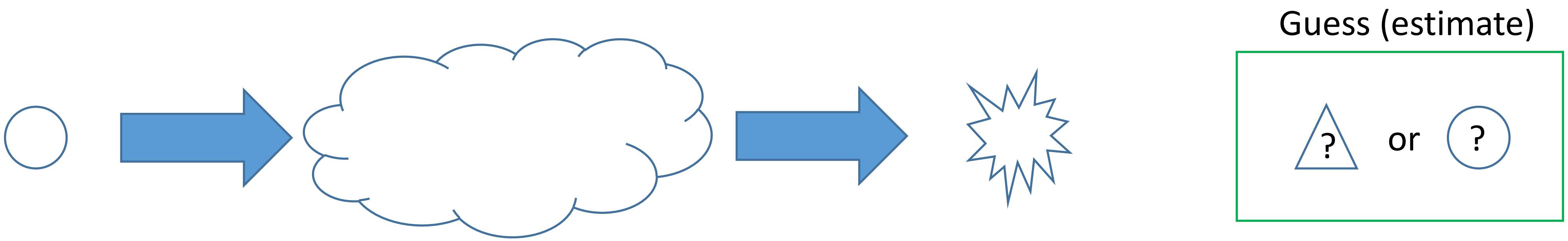
Signals in Noise

- Consider a transmission systems sending circles and triangles (two symbols agreed between sender and receiver)
- If there is no noise or distortion, receiver should be easily able to tell what was sent



Signals in Noise

- If there is noise or distortion, the receiver has to guess (estimate), whether the sent symbol was a circle or a triangle



Signals in Noise

- When you guess (**estimate**), there is a chance (**probability**) of making a mistake (**error**)

Actually Sent	**Estimate** (what receiver thinks was sent)
○	○
○	△
△	○
△	△

Probability of Error

$$P_e = P(guess\ \triangle \mid sent\ \bigcirc) + P(guess\ \bigcirc \mid sent\ \triangle)$$

Clearly, we would like to keep the probability of error as low as possible (i.e., minimize P_e)

Signals in Noise

- Let's make the situation more interesting.
- Suppose sender informs that he will send circle 80% of the time
- Then it should be more important to avoid the error:

P(guess triangle | sent circle)

$$P(\bigcirc) = 0.8$$
$$P(\triangle) = 0.2$$

Already known (prior) probabilities of sending circles and triangles

Probability of Error

$$P_e = P(\bigcirc)P(guess\triangle \mid sent\bigcirc) + P(\triangle)P(guess\bigcirc \mid sent\triangle)$$

Let’s talk about the choice of Symbols

- Our choice of symbols clearly affects the probability of error P_e
- For instance, we have seen previously the following example

Probability of guessing “ta” when actually “za” is sent →

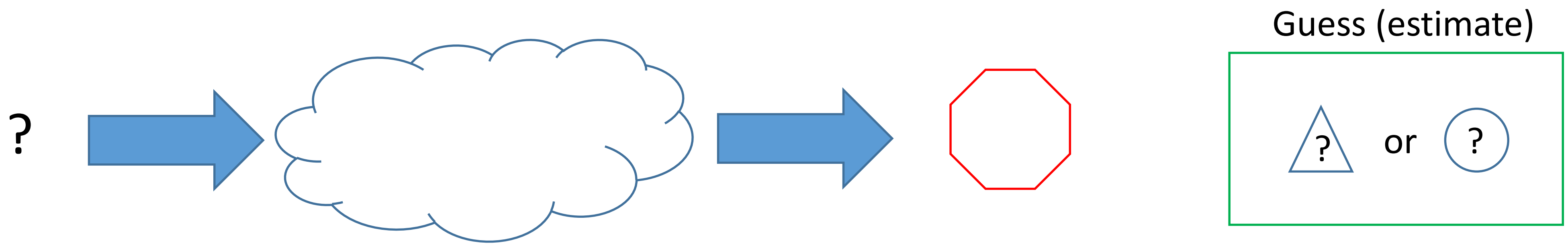
P (ط | ظ) = High!!

P (ط | ع) = Low!!

Lesson: perhaps we should not use symbols that are too similar (close in some sense) to each other!

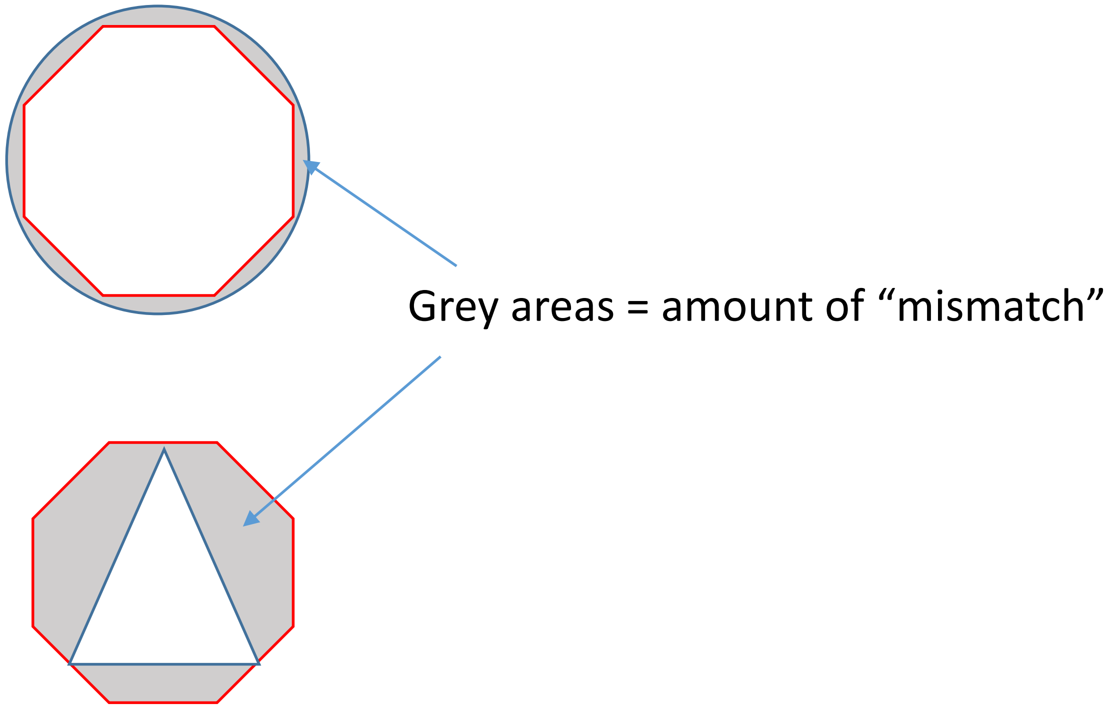
Let’s talk about the choice of Symbols

- Back to circles and triangles
- Suppose you receive an octagon. How do you decide if the sent symbol was a triangle or circle?



Let’s talk about the choice of Symbols

- Back to circles and triangles
- Suppose you receive decagon. How do you decide if the sent symbol was a triangle or circle?



Decision: Circle was sent
Reason: received symbol has less “mismatch” with circle compared to mismatch with triangle (i.e., received symbol is more **similar** to a circle than a triangle)

Q. How can we check "similarity" mathematically?

Answer: "Similarity" between two vectors can be checked using the inner product or the Euclidean distance between them

Inner product = projection of one vector (or signal) on other

$$\mathbf{s}_i^{\mathrm{T}}\mathbf{s}_k = \sum_{j=1}^{N} s_{ij}s_{kj}$$

Signals in vector form

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \qquad \mathbf{s}_k = \begin{bmatrix} s_{k1} \\ s_{k2} \\ \vdots \\ s_{kN} \end{bmatrix}$$

$$\int_0^T s_i(t)s_k(t)\,\mathrm{d}t$$

Signals in continuous-time form

Energy and Norm

The inner product of a signal with itself gives the energy (E_i) of the signal
(and equals the square of the norm)

$$E_i = \int_0^T s_i^2(t)\,dt$$

Signals in continuous-time form

$$E_i = \sum_{j=1}^{N} s_{ij}^2 = \|\mathbf{s}_i\|^2$$

Signal in vector form

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$$

Q. How can we check "similarity" mathematically?

Answer: "Similarity" between two vectors can be checked using the inner product or the Euclidean distance between them

Euclidean Distance = how far apart are the tips of the vectors

$$\|\mathbf{s}_i - \mathbf{s}_k\|^2 = \sum_{j=1}^{N} (s_{ij} - s_{kj})^2$$

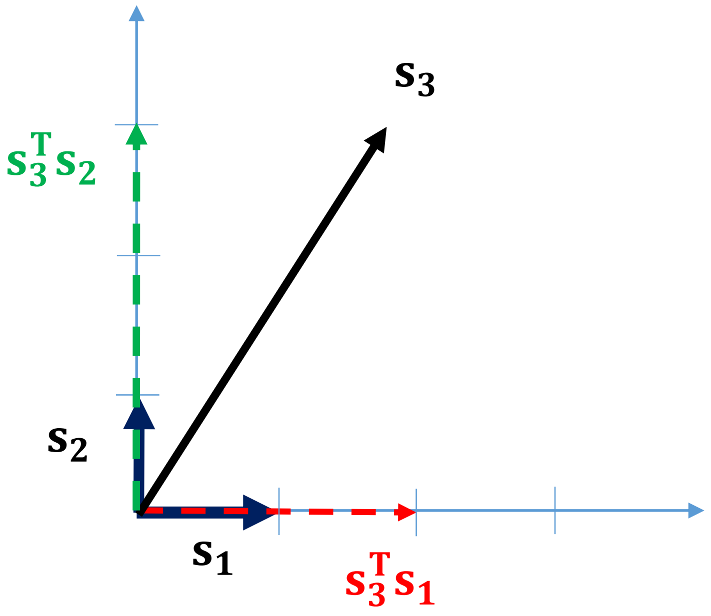
Signals in vector form

$$\int_0^T (s_i(t) - s_k(t))^2 \, dt$$

Signals in continuous-time form

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \qquad \mathbf{s}_k = \begin{bmatrix} s_{k1} \\ s_{k2} \\ \vdots \\ s_{kN} \end{bmatrix}$$

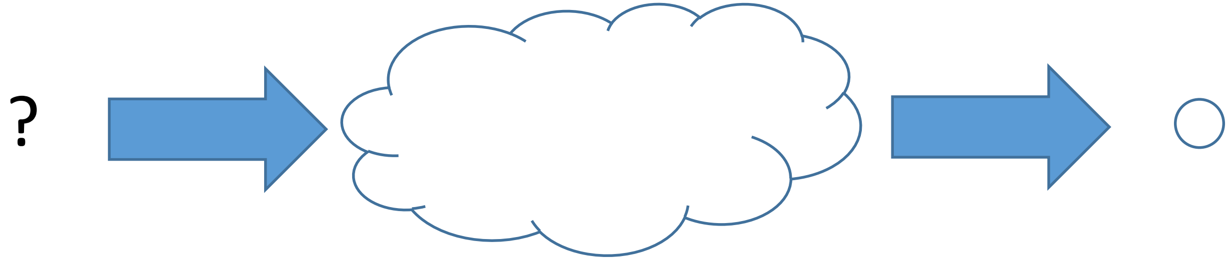
Examples



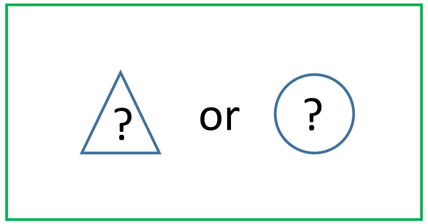
- Consider three vectors
  - $\mathbf{s}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \ \mathbf{s}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \ \mathbf{s}_3 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$
- We can see that
  - $\mathbf{s}_1$ and $\mathbf{s}_2$ have no projection on each other (as their inner product $\mathbf{s}_1^{\mathrm{T}}\mathbf{s}_2 = 0$)
    - Thus, they are very “dis-similar”
  - $\mathbf{s}_3$ has projection on both $\mathbf{s}_1$ and $\mathbf{s}_2$ (inner products: $\mathbf{s}_3^{\mathrm{T}}\mathbf{s}_1 = 2, \ \mathbf{s}_3^{\mathrm{T}}\mathbf{s}_2 = 3$)
    - It has more “similarity” with $\mathbf{s}_2$ (since the inner product is higher)

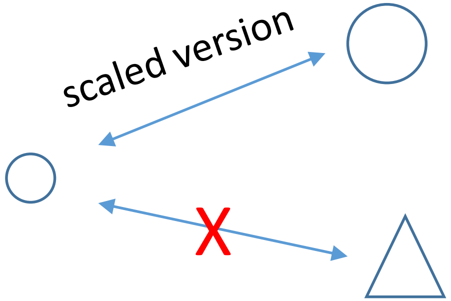
Back to the choice of Symbols …

- Suppose you receive small circle
  - You can easily guess that it is a scaled version of circle



Guess (estimate)



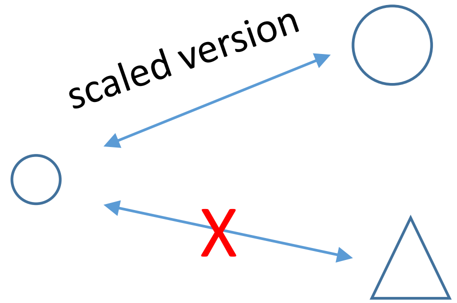


Side lesson: better avoid symbols which can be created by simple scaling of each other

Circle and Triangle are a good choice since neither one can be created by a simple scaling of the other

Orthogonality: “completely dis-similar”

Circle and Triangle are a good choice since neither one can be created by a simple scaling of the other (this makes decision-making easier at the receiver)



Q. How can we put such “complete dis-similarity” mathematically?

- We saw that “inner product” can be used as a measure of “similarity”
- So if the inner product turns out to be zero we can say that the vectors (or signals) are very dis-similar
- Mathematically, we say they are “***orthogonal***” to each other

Orthogonality

Two vectors (or signals) are said to be orthogonal to each other if their inner product is zero (i.e., each one has zero projection on the other)

If **Inner product = 0** then s_i and s_k are orthogonal to each other

$$\mathbf{s}_i^{\mathrm{T}}\mathbf{s}_k = \sum_{j=1}^{N} s_{ij}s_{kj} = 0$$

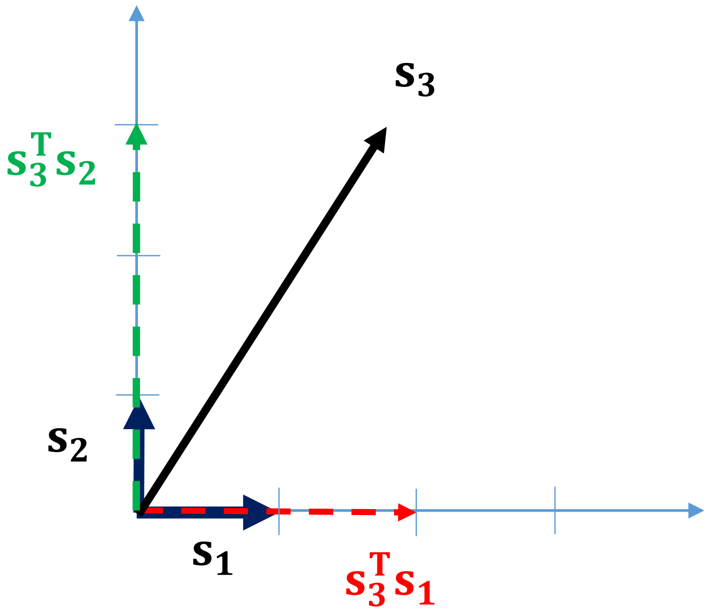
Signals in vector form

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \quad \mathbf{s}_k = \begin{bmatrix} s_{k1} \\ s_{k2} \\ \vdots \\ s_{kN} \end{bmatrix}$$

$$\int_0^T s_i(t)s_k(t)\,\mathrm{d}t = 0$$

Signals in continuous-time form

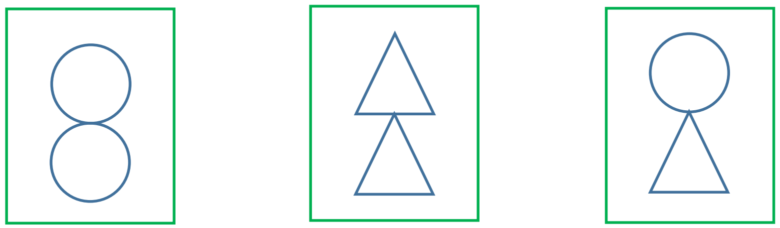
Examples



- Consider three vectors
  - $\mathbf{s}_1 = \begin{bmatrix}1\\0\end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix}0\\1\end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix}2\\3\end{bmatrix}$
- As we saw before, $\mathbf{s}_1^T\mathbf{s}_2 = 0$ (zero inner product)
  - Thus they are orthogonal to each other (no projection on each other)
- $\mathbf{s}_3$ has projection on both $\mathbf{s}_1$ and $\mathbf{s}_2$ (inner products: $\mathbf{s}_3^T\mathbf{s}_1 = 2$, $\mathbf{s}_3^T\mathbf{s}_2 = 3$)
  - Therefore, it is not orthogonal to either one

Finding Distinct Elements – Signal Space

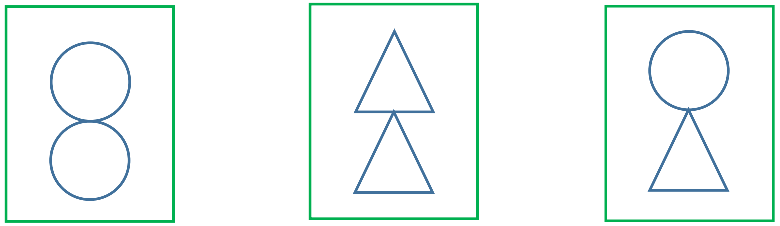
- Let’s make things more interesting …
- Let’s now consider three signals
- What are the minimum distinct (orthogonal) elements that can represent all three?



Signal A Signal B Signal C

Finding Distinct Elements – Signal Space

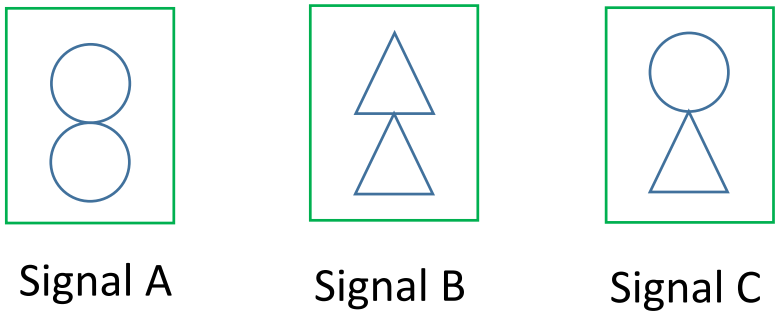
- What are the minimum distinct (orthogonal) elements that can represent all three signals below?
  - Clearly, these are: Circle and Triangle
- **Such mutually orthogonal (distinct) elements are called "bases"**



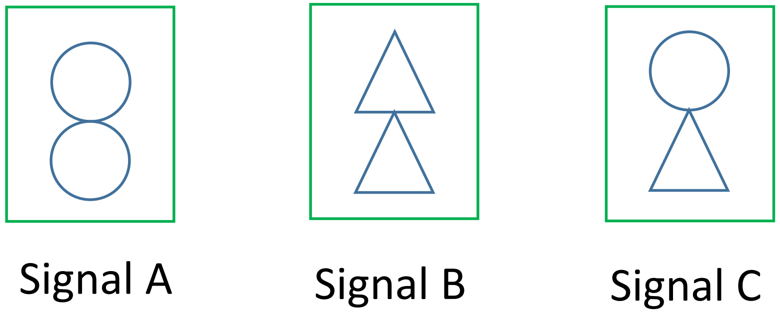
Signal A Signal B Signal C

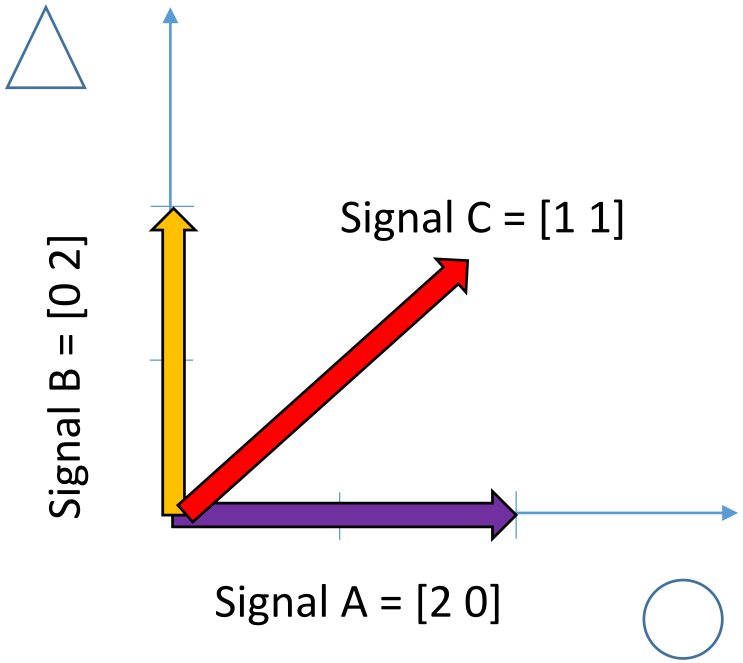
Geometric Representation of Signals

- We can "describe" the three signals in terms of two elements ("bases")
  - Signal A = [2 0] (i.e., it has two circles and no triangle)
  - Signal B = [0 2] (i.e., it has no circles, and two triangles)
  - Signal C = [1 1] (i.e., it has one circle and one triangle)
- **We can thus re-write the three signals as vectors**
  - **This is called signal space representation or geometric representation**



Bases, Signal Space, and Signal Vectors





Circle and Triangle are the "**bases**" (x and y axis), and can be written as **Circle = [1 0], Triangle = [0 1]**

The space spanned by the bases is the **signal space** (here it is the plane described by the two axes)

Signals written (or drawn) in vector form using the bases are called **signal vectors** (note that all three signal vectors lie in the signal space)

Putting Bases, Signal Space, and Signal Vectors Mathematically

$\phi_1(t),\ \phi_2(t),\ \ldots,\ \phi_N(t)$

N orthonormal basis functions

normalized to unit energy

$$\int_0^T \phi_i(t)\phi_j(t)\,\mathrm{d}t = \delta_{ij} = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

orthogonal to each other

Orthonormal = orthogonal and normalized.

Putting Bases, Signal Space, and Signal Vectors Mathematically

$$s_1(t),\ s_2(t),\ \ldots,\ s_M(t)$$

M signals in continuous-time form ($M \leq N$)

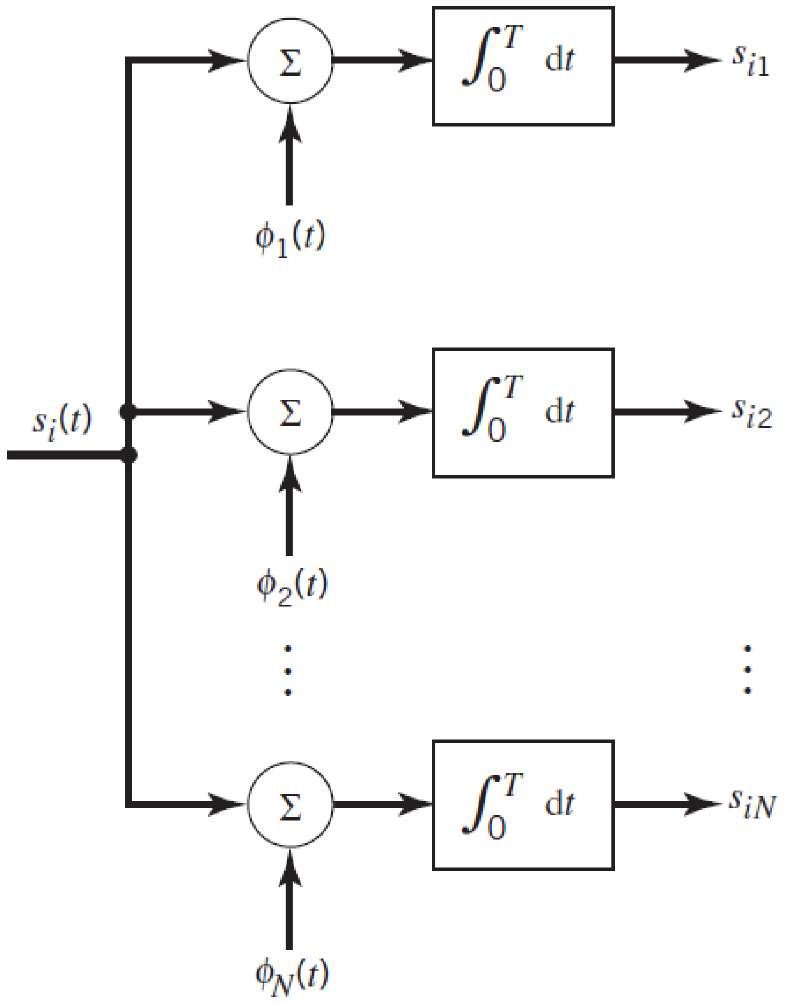
$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad \begin{cases} 0 \leq t \leq T \\ i = 1, 2, \ldots, M \end{cases}$$

Signals written in terms of the basis functions

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}, \qquad i = 1, 2, \ldots, M$$

M signals in signal space (vector) form

Putting Bases, Signal Space, and Signal Vectors Mathematically

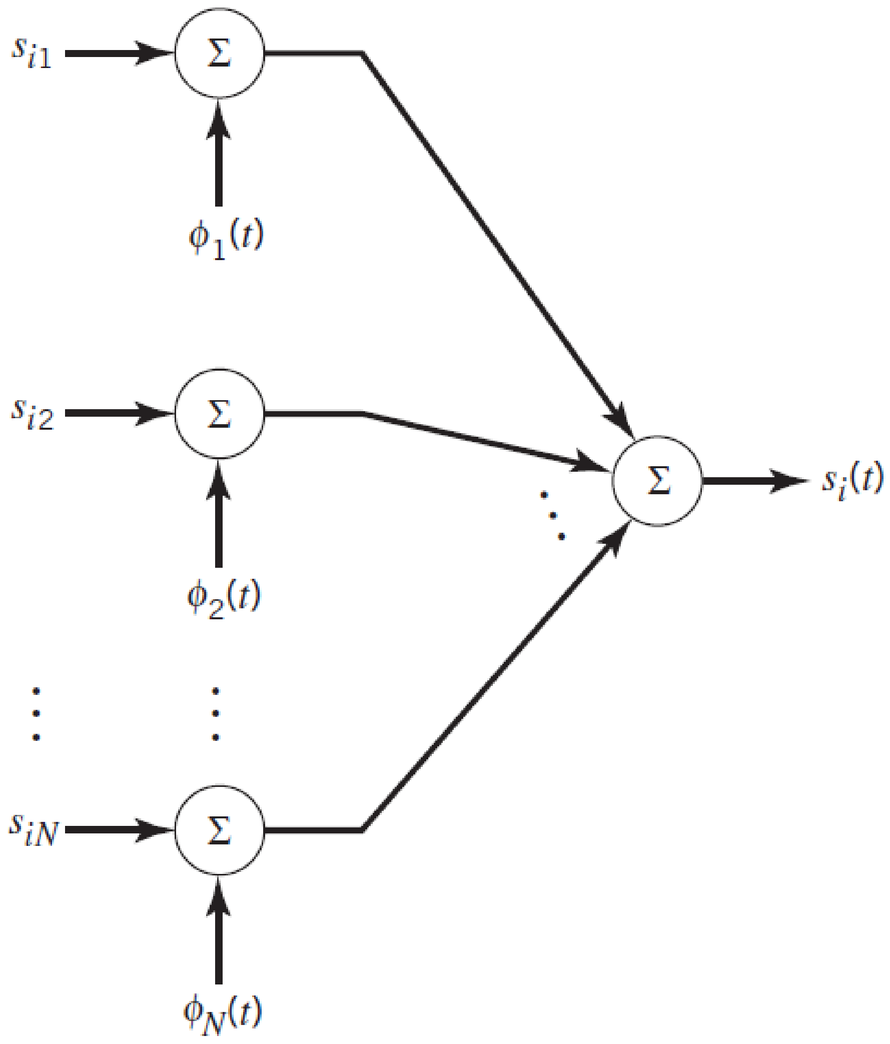


Formula for calculating vector values $\{s_{ij}\}$

$$s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,dt \qquad \begin{array}{l} i = 1, 2, \ldots, M \\ j = 1, 2, \ldots, N \end{array}$$

Interpretation: s_{ij} is the projection of signal $s_i(t)$ on basis function $\phi_j(t)$

Putting Bases, Signal Space, and Signal Vectors Mathematically



Knowing signal-space values $\{s_{ij}\}$ and the basis functions $\{\phi_1(t), \ldots, \phi_N(t)\}$ we can reconstruct the continuous-time signal $s_i(t)$

$$s_i(t) = \sum_{j=1}^{N} s_{ij}\phi_j(t), \qquad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

Interpretation: $s_i(t)$ can be written as a "weighted sum" of basis functions $\{\phi_1(t), \ldots, \phi_N(t)\}$

Example

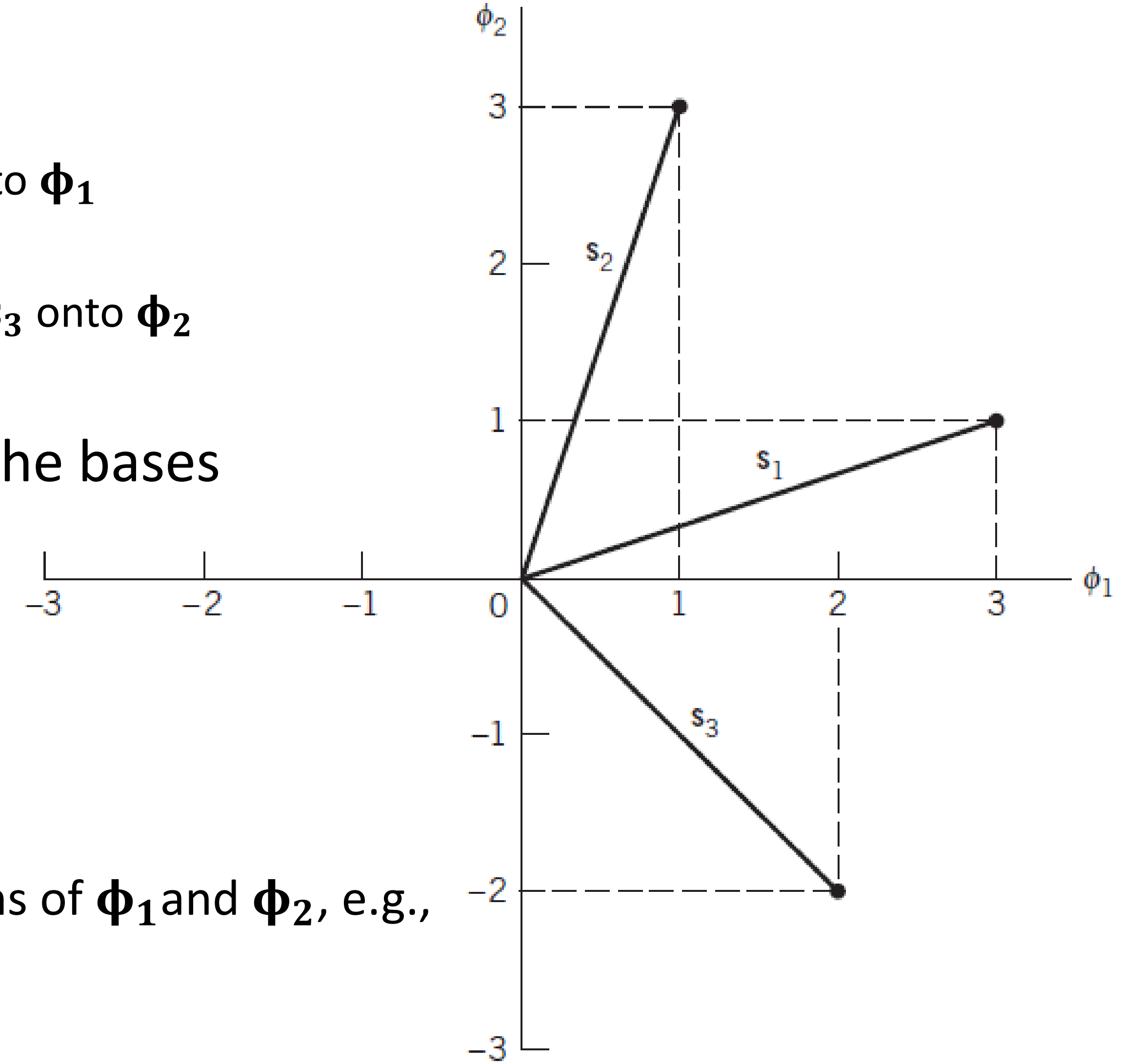
$\mathbf{s}_3^T \boldsymbol{\phi}_1$

Projection of $\mathbf{s}_3$ onto $\boldsymbol{\phi}_1$

- Consider three vectors
  - $\mathbf{s}_1 = \begin{bmatrix} 3 \\ 0 \end{bmatrix}$, $\mathbf{s}_2 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$, $\mathbf{s}_3 = \begin{bmatrix} 2 \\ -2 \end{bmatrix}$

Projection of $\mathbf{s}_3$ onto $\boldsymbol{\phi}_2$

- Clearly, these three can be represented by the bases
  - $\boldsymbol{\phi}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $\boldsymbol{\phi}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$
  - Are these bases orthonormal?
    - $\boldsymbol{\phi}_1^T \boldsymbol{\phi}_2 = 0$
    - $\boldsymbol{\phi}_1^T \boldsymbol{\phi}_1 = 1$, $\boldsymbol{\phi}_2^T \boldsymbol{\phi}_2 = 1$
    - Yes!
  - All three signals can be written as weighted sums of $\boldsymbol{\phi}_1$ and $\boldsymbol{\phi}_2$, e.g.,
    - $\mathbf{s}_3 = (2)\begin{bmatrix} 1 \\ 0 \end{bmatrix} + (-2)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 2\boldsymbol{\phi}_1 + (-2)\boldsymbol{\phi}_2$



Gram-Schmidt Orthogonalization Procedure

- We discussed basis functions $(\phi_1(t), \dots, \phi_N(t))$ for the geometric (vector) representation of signals
- Next question is **how can we find the basis functions for a given set of signals**?
- One useful procedure that helps us in this is the Gram-Schmidt Orthogonalization Procedure

Gram-Schmidt Orthogonalization Procedure

1. Given M signals $s_1(t),\ s_2(t),\ \ldots,\ s_M(t)$ with energies $E_i = \int_0^T s_i^2(t)\,dt$

2. Choose one arbitrarily, and get the first basis function as

$$\phi_1(t) = \frac{s_1(t)}{\sqrt{E_1}}$$

$s_2(t)$ ***minus*** the part of $s_2(t)$ that can be written in terms of $\phi_1(t)$

3. Get the second basis functio as

$$\phi_2(t) = \frac{g_2(t)}{\sqrt{\int_0^T g_2^2(t)\,dt}}$$

where

$$g_2(t) = s_2(t) - s_{21}\phi_1(t)$$

$$s_{21} = \int_0^T s_2(t)\phi_1(t)\,dt$$

Gram-Schmidt Orthogonalization Procedure

4. In general, get the i-th basis function as

$$\phi_i(t) = \frac{g_i(t)}{\sqrt{\int_0^T g_i^2(t)\,dt}},$$

where

$$g_i(t) = s_i(t) - \sum_{j=1}^{i-1} s_{ij}\phi_j(t)$$

$$i = 1, 2, \ldots, N \quad (N \leq M)$$

$$s_{ij} = \int_0^T s_i(t)\phi_j(t)\,dt, \qquad j = 1, 2, \ldots, i-1$$

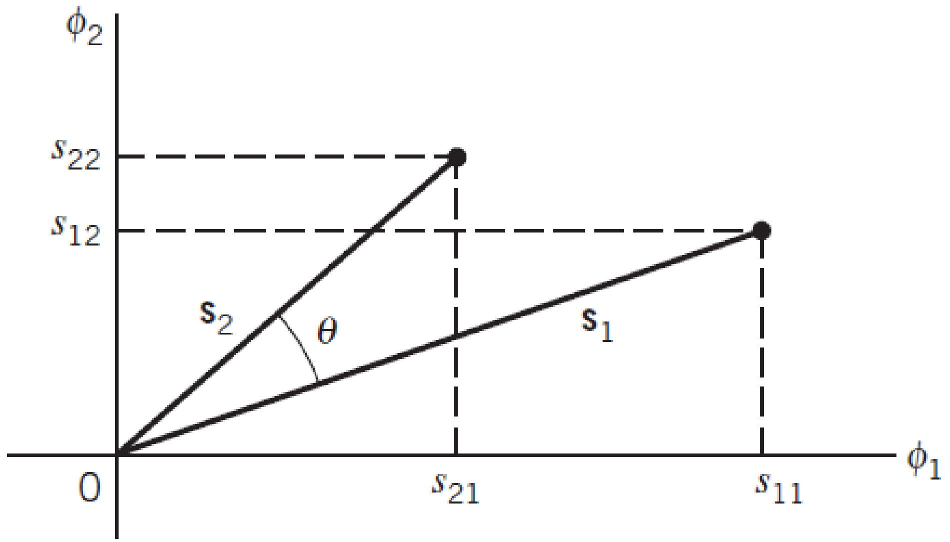
Other useful relations

Angle Between Vectors

$$\cos\theta = \frac{\mathbf{s}_1^T \mathbf{s}_2}{\|\mathbf{s}_1\|\|\mathbf{s}_2\|}$$

$$= \frac{\int_{-\infty}^{\infty} s_1(t)s_2(t)\,dt}{\left(\int_{-\infty}^{\infty} s_1^2(t)\,dt\right)^{1/2}\left(\int_{-\infty}^{\infty} s_2^2(t)\,dt\right)^{1/2}}$$

Example



Other useful relations

Schwarz Inequality

$$\left| \int_{-\infty}^{\infty} s_1(t) s_2^*(t)\, \mathrm{d}t \right| \le \left(\int_{-\infty}^{\infty} |s_1(t)|^2\, \mathrm{d}t \right)^{1/2} \left(\int_{-\infty}^{\infty} |s_2(t)|^2\, \mathrm{d}t \right)^{1/2}$$

Questions?? Thoughts??

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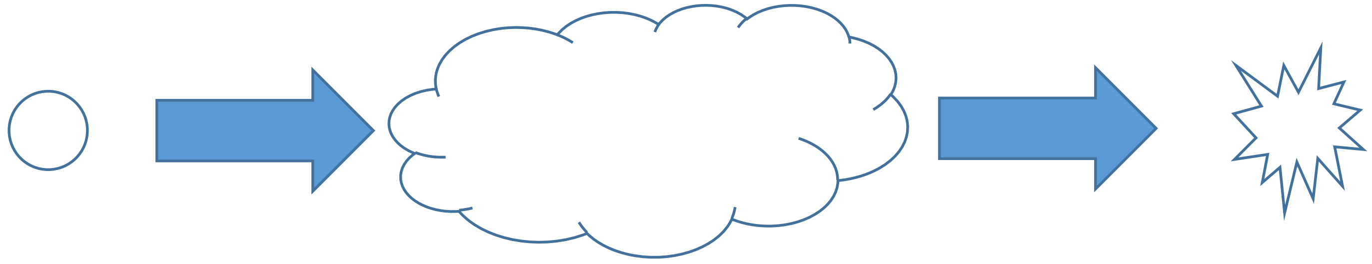
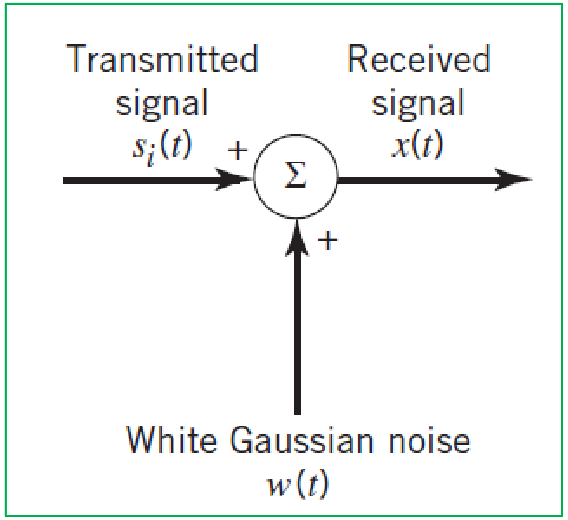
AWGN – the simplest channel

- When signals pass through transmission mediums (“channels”), they often suffer noise and other distortions (*changes*)
- We use several different models to represent such channel effects on signals



AWGN – the simplest channel

- The simplest model we have for channel effects is the AWGN model
  - A = Additive
  - W = White
  - G = Gaussian
  - N = Noise



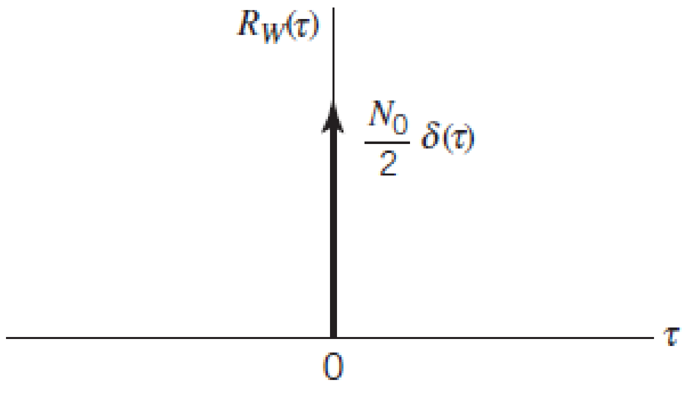
$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

Revision – what is White Gaussian Noise?

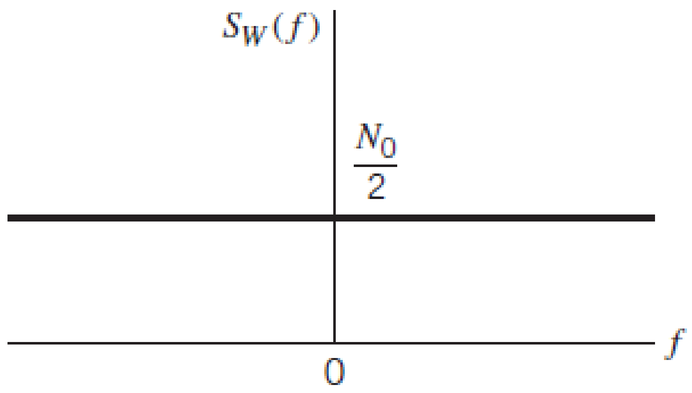
- Noise
  - Something **unwanted**
- White
  - A (stationary) process so random that there is **no correlation between any two samples**!
  - White Noise is a **good approximation** of combined effect of many unrelated noise sources
- Gaussian
  - A white noise process that is characterized by the Gaussian distribution is called **White Gaussian Noise**

Autocorrelation and Power Spectral Density (PSD) of a white noise process

- A white noise process has equal contribution from *all* the frequencies (just like white light has equal contribution from all the visible frequencies)



Time description: **No correlation** between any two samples



Frequency description: **equal** contribution from **all frequencies**

AWGN - mathematically

Continuous-time form

$$x(t) = s_i(t) + w(t), \quad \begin{cases} 0 \le t \le T \\ i = 1, 2, \ldots, M \end{cases}$$

Signal-Space (vector) form

$$\mathbf{x} = \mathbf{s}_i + \mathbf{w}, \qquad i = 1, 2, \ldots, M$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \cdot \\ \cdot \\ \cdot \\ x_N \end{bmatrix} \qquad \mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} w_1 \\ w_2 \\ \cdot \\ \cdot \\ \cdot \\ w_N \end{bmatrix}$$

Continuous-Time → Signal-Space

$$s_{ij} = \int_0^T s_i(t)\,\phi_j(t)\,\mathrm{d}t$$

$$w_j = \int_0^T w(t)\,\phi_j(t)\,\mathrm{d}t$$

$$\phi_1(t),\ \phi_2(t),\ \ldots,\ \phi_N(t)$$

N orthonormal basis functions

AWGN - Statistically

$$\mathbf{X} = \mathbf{s}_i + \mathbf{W}$$

Gaussian

Considered deterministic (non-random)

Zero-Mean, White, Gaussian

Statistical Properties of $\mathbf{s}_i$

$$\mathbf{s}_i = \begin{bmatrix} s_{i1} \\ s_{i2} \\ \vdots \\ s_{iN} \end{bmatrix}$$

Mean $\mu_{s_{ij}} = E[s_{ij}] = s_{ij}$

Variance $\sigma^2_{s_{ij}} = Var[s_{ij}] = 0$

Since $\mathbf{s}_i$ is considered deterministic (non-random)

Statistical Properties of $\mathbf{W}$

$$\mathbf{W} = \begin{bmatrix} W_1 \\ W_2 \\ \cdot \\ \cdot \\ \cdot \\ W_N \end{bmatrix}$$

Zero-Mean $\quad \mu_{W_j} = E[W_j] = 0$

White $\quad R_W(\tau) = \begin{cases} \frac{N_0}{2}, & \tau = 0 \\ 0, & \tau \neq 0 \end{cases}$ $\Rightarrow$ $\begin{cases} \sigma^2_{W_j} = Var[W_j] = \frac{N_0}{2} \\ E[W_j W_k] = 0 \quad i \neq k \\ PSD(W) = \frac{N_0}{2} \end{cases}$

Gaussian $\quad f_{W_j}(w_j) = \frac{1}{\sqrt{\pi N_0}} \exp[-\frac{1}{N_0} w_j^2]$

White Gaussian $\quad f_{jk}(w_j, w_k) = f_{W_j}(w_j) f_{W_k}(w_k), \qquad j \neq k$

$N_0 = some\ constant$

Statistical Properties of $\mathbf{X}$

$$\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \\ \cdot \\ \cdot \\ \cdot \\ X_N \end{bmatrix}$$

Mean $\mu_{X_j} = E[X_j] = s_{ij}$

Variance & Covariance

$$\sigma_{X_j}^2 = Var[X_j] = \frac{N_0}{2}$$

$$Cov[X_j X_k] = 0 \quad i \neq k$$

Gaussian

$$f_{X_j}(x_j | m_i) = \frac{1}{\sqrt{\pi N_0}} \exp\left[-\frac{1}{N_0}(x_j - s_{ij})^2\right]$$

Uncorrelated Gaussian

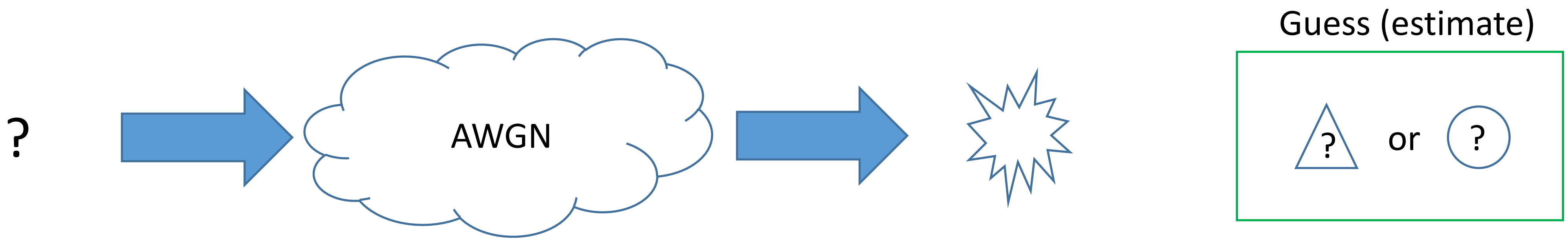
$$f_{\mathbf{X}}(\mathbf{x} | m_i) = \prod_{j=1}^{N} f_{X_j}(x_j | m_i) = (\pi N_0)^{-N/2} \exp\left[-\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2\right]$$

m_i = symbol represented by signal $\mathbf{s}_i$

Evaluated based on statistical properties of $\mathbf{s}_i$ and $\mathbf{W}$

Decoding the Received Signal

- Next, we are interested in finding ways of deciding which symbol was sent



Decoding the Received Signal

- **Assumptions**
  - Channel: AWGN
  - Symbols: $m_1, m_2, \ldots, m_M$
  - Signal used to represent symbol m_i (in vector form) : $\mathbf{s}_i$
  - Received signal vector: $\mathbf{x} = \mathbf{s}_i + \mathbf{w}$
- **Problem statement**
  - Given observation vector $\mathbf{x}$ and the above assumptions, decide (estimate) which of the symbols $(m_1, m_2, \ldots, m_M)$ was sent!!
- **Notation**
  - We shall denote our estimate (decision) as $\hat{m}$

Maximum A Posteriori (MAP) Decoder

MAP Decoder

Prior probability of transmitting symbol m_k , i.e.,
$\pi_k = \mathrm{P}(m_k)$

- **Algorithm**

Conditional probability of seeing $\mathbf{x}$ given/assuming symbol m_k was sent

1. The cost function is

$$\pi_k f(\mathbf{x}|m_k)$$

2. Calculate value of the cost function for each symbol $\{m_k\}_{k=1,\ldots,M}$
3. Decide $\widehat{m} = m_i$ if the cost function is maximum at $k = i$

Maximum Likelihood (ML) Decoder

Special case of MAP Decoder

When all symbols have equal probability of transmission ($\pi_1 = \pi_2 = \cdots = \pi_M$)

- **Assumptions for MAP Decoder**
  - Symbols: $m_1, m_2, \dots, m_M$
  - Signal used to represent symbol m_i (in vector form) : $\mathbf{s}_i$
  - Received signal vector: $\mathbf{x} = \mathbf{s}_i + \mathbf{w}$
- **Additional Assumption for ML Decoder**
  - All symbols have equal probability of being transmitted, i.e.,
    $\pi_1 = \pi_2 = \cdots = \pi_M$

Background: How do we get to the MAP decoder?

Likelihood Function

Probability that observation x was generated by symbol m_k (i.e., the conditional probability seen from the receiver's point of view)

$$l(m_k) \triangleq f(\mathbf{x}|m_k)$$

Log-Likelihood Function

$\mathbf{ln} = \boldsymbol{natural\ log}$

We often find it more convenient to work with the log-likelihood function

$$L(m_k) \triangleq \ln f(\mathbf{x}|m_k)$$

ML Decoder

- **Algorithm**

log-likelihood, i.e., $L(m_k) = \ln f(\mathbf{x}|m_k)$

1. The cost function is

$$L(m_k)$$

2. Calculate value of the cost function for each symbol $\{m_k\}_{k=1,\dots,M}$
3. Decide $\hat{m} = m_i$ if the cost function is maximum at $k = i$

Background: How do we get to the ML decoder?

Decision Regions

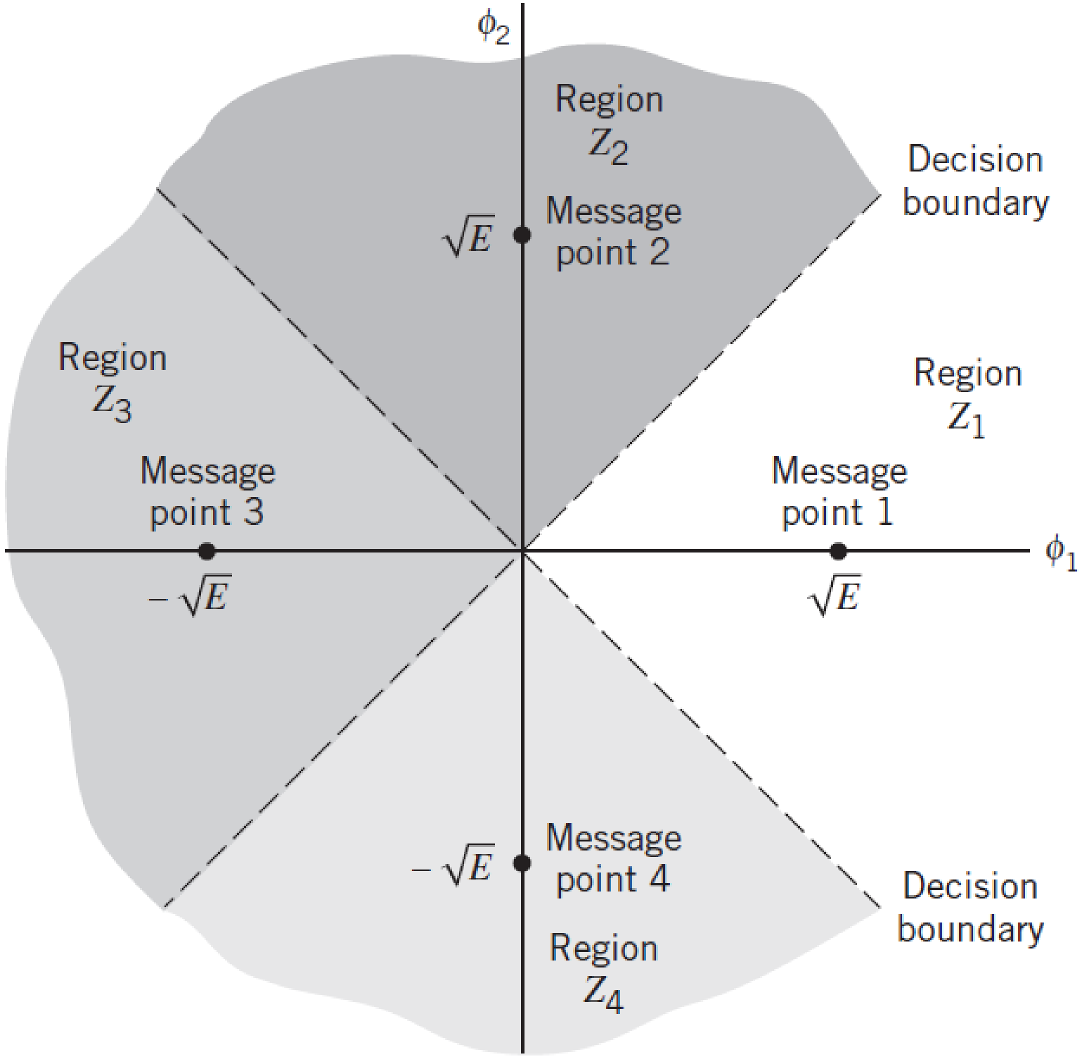
Symbol m_i ⟷ Region Z_i

It is useful to have a graphical interpretation of the maximum likelihood decision rule. Let Z denote the N-dimensional space of all possible observation vectors $\mathbf{x}$. We refer to this space as the *observation space*. Because we have assumed that the decision rule must say $\hat{m} = m_i$, where $i = 1, 2, \ldots, M$, the total observation space Z is correspondingly partitioned into *M-decision regions*, denoted by $Z_1, Z_2, \ldots, Z_M$.

In short, if vector x lies in region Z_i, we decide that the sent symbol was m_i (and vice-versa)

Decision Regions

Symbol m_i ⟷ Region Z_i



Illustrating the partitioning of the observation space into decision regions for the case when $N = 2$ and $M = 4$; it is assumed that the M transmitted symbols are equally likely.

ML Rule written in terms of Decision-Regions

Observation vector $\mathbf{x}$ lies in region Z_i if $L(m_k)$ is maximum for $k = i$.

ML Decoder for AWGN (Formulation 1)

Observation vector $\mathbf{x}$ lies in region Z_i if $\sum_{j=1}^{N} (x_j - s_{kj})^2$ is minimum for $k = i$.

Background: How do we get Formulation 1?

It can be shown that for AWGN case, the log-likelihood function is (ignoring some constant terms)

$$L(m_i) = -\frac{1}{N_0}\sum_{j=1}^{N}(x_j - s_{ij})^2, \qquad i = 1, 2, \ldots, M$$

Clearly, this function is maximized when the summation is minimized!!

ML Decoder for AWGN (Formulation 2)

Observation vector $\mathbf{x}$ lies in region Z_i if Euclidean distance $\|\mathbf{x} - \mathbf{s}_k\|$ is minimum for $k = i$

Background: How do we get Formulation 2?

We simply note that the summation in Formulation 1 is just the square of the Euclidean Distance between the observation vector $\mathbf{x}$ and the transmitted signal vector $\mathbf{s}_k$

$$\sum_{j=1}^{N} (x_j - s_{kj})^2 = \|\mathbf{x} - \mathbf{s}_k\|^2$$

Interpretation: so for AWGN case, the ML rule simply says "choose (decide in favor of) the message point (symbol) closest to the received signal point (observation)"

ML Decoder for AWGN (Formulation 3)

Observation vector $\mathbf{x}$ lies in region Z_i if $\left(\sum_{j=1}^{N} x_j s_{kj} - \frac{1}{2}E_k\right)$ is maximim for $k = i$, where E_k is transmitted energy.

$$E_k = \sum_{j=1}^{N} s_{kj}^2$$

Background: How do we get Formulation 3?

We expand the summation of Formulation 2 and note that only the second two terms depend on the optimization index k (so the first term can be dropped from the optimization)

$$\sum_{j=1}^{N}(x_j - s_{kj})^2 = \sum_{j=1}^{N} x_j^2 - 2\sum_{j=1}^{N} x_j s_{kj} + \sum_{j=1}^{N} s_{kj}^2$$

How can we use the ML Decoder?

We saw that all three formulations of the ML decoder require the observation vector $\mathbf{x}$**.** Depending on how we produce $\mathbf{x}$ from the time observation $x(t)$**,** we can have different receivers, two of which are **Correlation Receiver** and **Matched Filter Receiver**.

Matched Filter?

Given a pulse signal $\phi(t)$ occupying the interval $0 \le t \le T$, a linear time-invariant filter is said to be matched to the signal $\phi(t)$ if its impulse response $h(t)$ satisfies the condition

$$h(t) = \phi(T-t) \qquad \text{for } 0 \le t \le T$$

Obtaining $\mathbf{x}$ through Matched Filters

$$x_j = \int_0^T x(t) h_j(T-t)\, \mathrm{d}t$$

Where h_j are the "matched" filters:

$$h_j(t) = \phi_j(T-t), \qquad \text{for } 0 \le t \le T \text{ and } j = 1, 2, \ldots, M$$

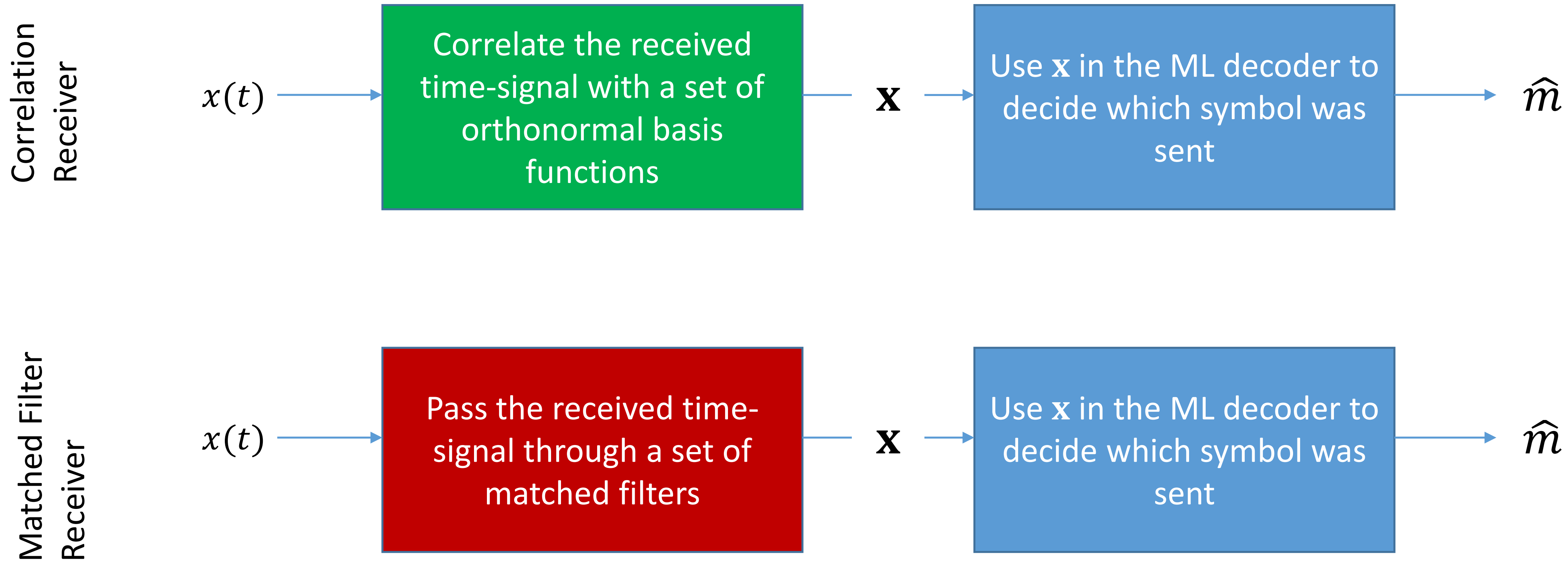
Background: How do we get the matched filter formulation?

A simple comparison of the correlator output with the output of a an LTI filter, gives the matched filter formulation

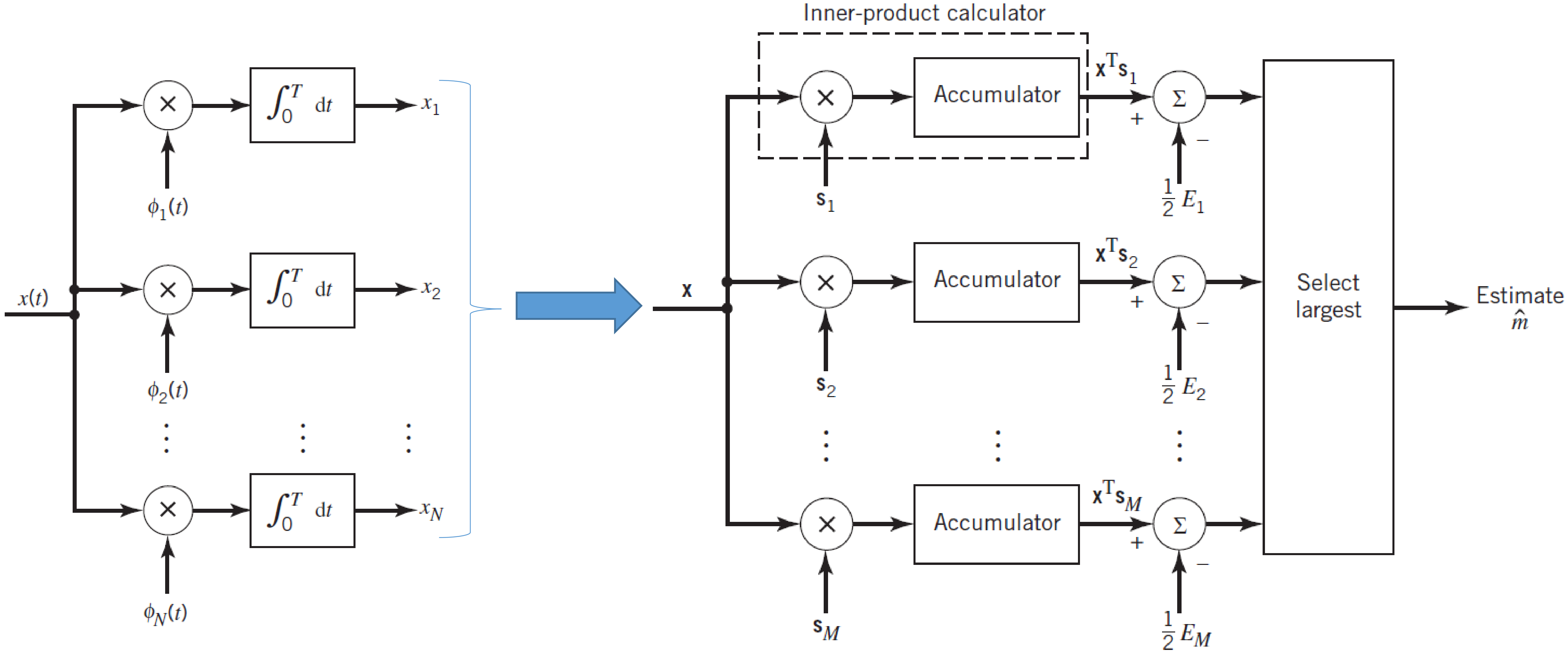
$$x_j = \int_0^T x(t)\phi_j(t)\,\mathrm{d}t \quad \equiv \quad y_j(T) = \int_0^T x(t)h_j(T-t)\,\mathrm{d}t$$

$$h_j(t) = \phi_j(T-t)$$

Correlation Receiver vs. Matched Filter Receiver



Correlation Receiver



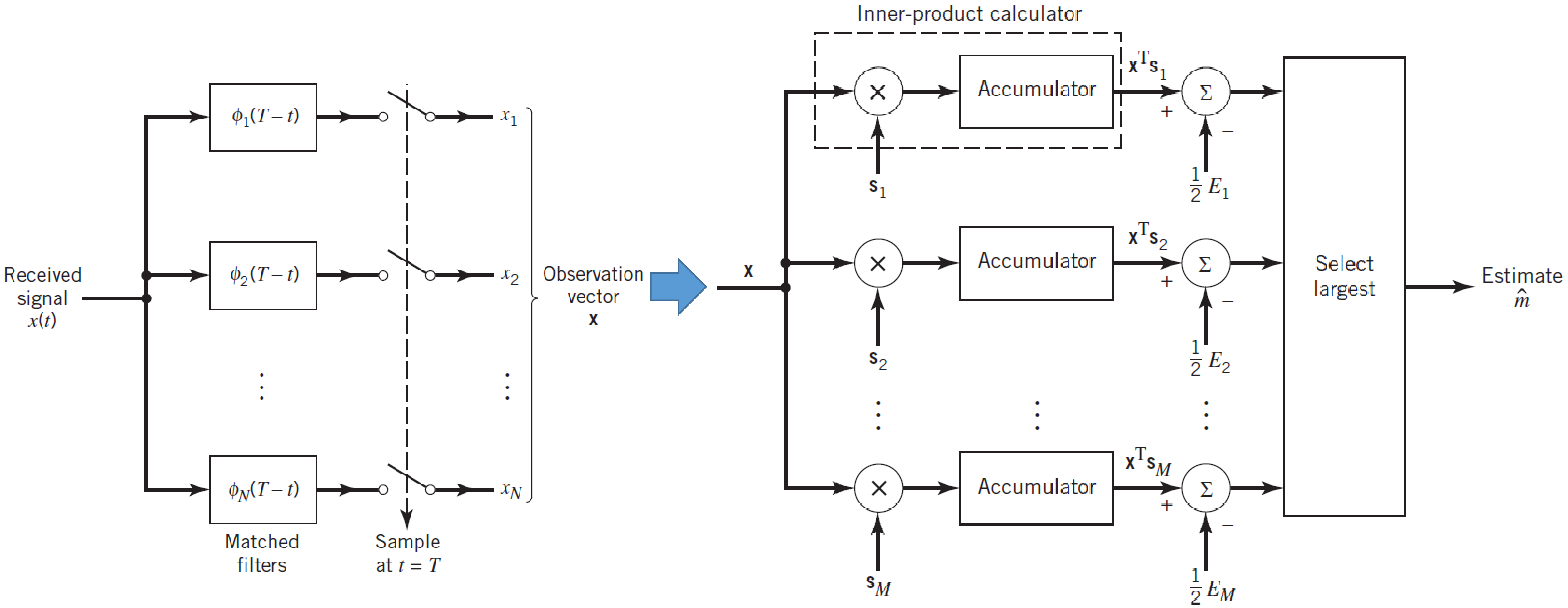
Correlation Receiver (*mathematically*)

Step 1: Find observation vector

$$x_j = \int_0^T x(t)\,\phi_j(t)\,\mathrm{d}t \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_N \end{bmatrix}$$

Step 2: Use ML Decoder (Formulation 1, 2, or 3) to find $\hat{m}$

Matched Filter Receiver



Matched Filter Receiver (*mathematically*)

Step 1: Find observation vector

$$x_j = \int_0^T x(t) h_j(T-t)\, \mathrm{d}t$$

$$h_j(t) = \phi_j(T-t)$$

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ . \\ . \\ . \\ x_N \end{bmatrix}$$

Step 2: Use ML Decoder (Formulation 1, 2, or 3) to find $\hat{m}$

Questions?? Thoughts??